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The Theoretical Wave Drag at Zero Lift of Fully-tapered Swept Wings of Arbitrary Section

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SUMMARY

An expression is deduced for the wave drag of a fully tapered swept wing of arbitrary section in the convenient form of a double integral involving the variation of wing-surface slope. It is concluded, in the general case, that the drag may best be computed by numerical integration, the method for which will be the subject of a further report.

In certain cases, however, the drag may be deduced for simple sections by direct integration, notably for the condition that $\sqrt{M^2-1}\cot\Lambda_{\frac{1}{2}}$ is small (where $\Lambda_{\frac{1}{2}}$ is the half-chord sweepback). Some results are quoted for this condition. Calculations suggest that the results thus obtained are approximately valid for $\sqrt{M^2-1}\cot\Lambda_{\frac{1}{2}} < 0.3$, and $A\tan\Lambda_{\frac{1}{2}} > 2\frac{1}{2}$. It is shown that, in this range, the relative drag of different wing sections, with the same root thickness and the same sweep on the maximum thickness line, is roughly that for the infinite wing (which has been deduced elsewhere); except that there is some rise (or fall) in the drag with decrease in aspect ratio A, due to the presence of a convex (or concave) curvature aft of the maximum thickness.

A further result deduced is that the drag of a fully tapered wing tends to a lower limit than that of an untapered wing, as the aspect ratio becomes infinitely large.

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List of Symbols

a	$= \lim_{k\to 0}\left\{Z'(k).k^{\frac{1}{2}}\right\}$, in Appendix III only.
a_n	Coefficients of Fourier expansion for $z'(k)$ - see equation (18).
c	Wing root chord.
$g(\xi, y, m)$	defined by equation (3) of Appendix I.
$h(\lambda, \kappa, \eta)$	defined by equation (12) of Appendix I.
j	defined by equation (1) of Appendix I.
j'	defined by equation (12) of Appendix I.
k	$= \xi/c$, where ξ is defined as in Appendix I.
ℓ	$= \left[x - \frac{y}{m(0)}\right] / \left[c\left(1-\frac{y}{s}\right)\right]$.
m	cotangent of angle of sweep.
$m(a)$	cotangent of angle of sweep of spanwise line which is placed at a fraction $\frac{a}{c}$ of chord.
n	maximum thickness position as fraction of chord.
Δp	perturbation pressure.
q	$= \frac{1}{2}\rho U^2$
r	radius of curvature of wing leading edge at root section.
s	wing semi-span.
t	maximum wing thickness (of root section)
w	upwash perturbation velocity
x, y, z	system of cartesian co-ordinates with origin at wing apex, with the positive x-axis lying downstream of apex along wing root, and the plane of the wing being $z = 0$.
$z(x)$	the wing semi-ordinate at the root section at a distance x from the apex.
A	wing aspect ratio ($= 4s/c$)
D	wing wave drag at zero lift.
$H(a)$	$= 0$ if $a < 0$; $= 1$ if $a \geq 0$.
$I(\lambda, \kappa)$	defined by equation (15) of Appendix I.
$I_1(\lambda, \kappa), I_2(\lambda, \kappa)$	defined by equation (22) of Appendix I.
$I_3(\lambda, \kappa)$	defined by equation (3) of main text.
$J(\eta)$	defined by equation (16) of Appendix I.
M	free-stream Mach Number.
$P(\lambda, \kappa)$	defined by equation (6) of main text.
$Q(\lambda, \kappa)$	defined by equation (9) of main text.
$\mathcal{R}\{\ \}$	'the real part of $\{\ \}$'.
S	Wing area ($=$ sc)
U	free-stream velocity
X	defined by equation (12) of Appendix I.
$Z(k)$	$= \frac{1}{2t}\, z(kc)$

$/\Lambda_\varepsilon\ldots$

Λ_a	sweepback of the spanwise line which is a fraction a of the chord, $= \cot^{-1}\left[m(ac)\right]$.
α	$= \beta s/c$
β	$= \sqrt{M^2-1}$
ε	used variously for a small quantity, and also for source plane strength (Appendix I)
η	$= y/s$
θ	$= \cos^{-1}(2k-1)$
κ	$= \sigma - k$
λ	$= \sigma - \ell$
μ	$= m\beta$
μ_a	$= \beta m(ac)$
ξ	$= \cosh^{-1}(2\sigma-1)$ in main text, and Appendix III, also used as a variable of integration in Appendix I, being the distance aft of the wing apex of the elementary source plane.
ρ	air density in free-stream
σ	$= s/cm(0)$
ϕ	$= \cos^{-1}(2\ell-1)$.

Primes denote differentiations with respect to the argument of functions.

N.B. Accepted convention of signs and principle parts :

$\sqrt{x} = i\left|\sqrt{|x|}\right|$ if $x < 0$,

$\ell n(x+iy) = \frac{1}{2}\ell n\left|x^2+y^2\right| + i\tan^{-1}\left(\frac{y}{x}\right)$, where $\pi \geqslant \tan^{-1}\left(\frac{y}{x}\right) > 0$ if $y \geqslant 0$.

$\cosh^{-1}x = \ell n(x+\sqrt{x^2-1})$.

$0 \leqslant \cos^{-1}x \leqslant \pi$, if $|x| \leqslant 1$.

1. Introduction.

At the present time the greatest need in assessing the theoretical wave drag of swept wings at supersonic speeds, is for some yardstick by which we may assess the effect of wing-section upon drag. Numerical results are available from linearized theory for wings of double-wedge section for a variety of planforms, and for untapered wings of biconvex section[1]. However most of the swept wings envisaged for supersonic aircraft have round-nosed sections, and apart from a single solution existing for wings of infinite span[2], no theoretical information is available for the drag of wings with such sections. This solution is of some value in suggesting the drag of untapered wings of finite aspect-ratio, because the bulk of the drag at low supersonic speeds derives from the flow near the root, but if applied to tapered wings one cannot properly account for the interaction of the varying effective angle of sweepback over the section on the total drag.

It is known for instance that for fully-tapered wings of double-wedge section the drag is reduced at low supersonic speeds by putting the maximum thickness line forward, so that it is more highly swept[3]. Whilst, of course, such a step would not be envisaged without taking into account the effects of such an alteration of the section upon the other characteristics of the wing, nevertheless it appears to be an important consideration in wing design.

However there has as yet been no method suggested for computing the drag of tapered wings of other sections than the double-wedge, except by the rather laborious process of approximating to the section shape by a polygon[4]. Our first concern here is then to formulate the problem and find the expression for the drag of a tapered wing assuming a completely arbitrary section.

We do this here for a fully-tapered wing, with a geometrically similar section at each chordwise station along the span. For the purposes of the discussion the slope of the section is assumed continuous, although we shall afterwards discuss the validity of the results if the section slope is discontinuous.

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The answer may be written down as a double integral, involving of course an expression for the variation of the section slope, and including parameters which relate to the wing planform and free-stream Mach Number. The integrand is a lengthy expression, and even if the wing section could be expressed by a simple formula, it seems doubtful whether in the general case the integrations could be performed to yield a value for the drag in terms of known functions of the planform and Mach Number parameters.

In view of this consideration, and in view of the additional fact that very few wing sections can be expressed adequately even by means of a complicated formula, it is desirable to consider a numerical method of integration; this will be the subject of a later report. In two particular cases, however, direct integration is possible for wings with simple section shapes (including some with bluff noses). One case is that of the wing of infinite span, and this has already been studied; the other case is the one which will be considered in some detail here, and which we shall call the application to 'slender wings' or, in other words, wings of high sweepback.

To be more precise it is shown in para. 2.4 that if A is the wing aspect ratio and $\Lambda_{\frac{1}{2}}$ is the half chord sweepback angle, then for small values of the parameter $\mu = \sqrt{M^2-1} \cot \Lambda_{\frac{1}{2}}$, the expression for the drag may be developed as a power series in μ^2 of the form

$$f_0(A \tan \Lambda_{\frac{1}{2}}) + f_1(A \tan \Lambda_{\frac{1}{2}})\mu^2 + \ldots\ldots$$

where the functions f_0, f_1, ... of $(A \tan \Lambda_{\frac{1}{2}})$ are bounded provided the wing trailing edge is sweptback (i.e. the value of $A \tan \Lambda_{\frac{1}{2}}$ exceeds 2). Thus for highly swept, or 'slender', wings (where $\cot \Lambda_{\frac{1}{2}} \to 0$), the expression for the drag reaches a finite limiting value which is a function of $A \tan \Lambda_{\frac{1}{2}}$; this limiting value is at least a good approximation even if μ is not zero but merely small. The same limiting value apparently applies if $M \to 1$, but of course it must be borne in mind that the linear theory is not strictly applicable to this condition. In fact, the same kind of restrictions apply to this approximation as are involved in the development of the well known 'low aspect ratio' or 'slender wing' theories of the lifting wing developed by R.T.Jones and others.

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The expression for the drag in this limiting condition is again a double integral, but is considerably simpler than the general expression to evaluate, so that a fairly wide numerical survey of the drag of various wing sections can reasonably be undertaken. The results of this 'slender wing' theory can be used to reach certain conclusions about the relative drag of various sections over a range of aspect ratio, and provide a check on our results in relation to data already published, as well as enabling subsequently an assessment of the accuracy of the processes of numerical integration to be made.

2. The Expression for the Drag.

The details of the mathematical argument will be found in Appendix I. Briefly the method is first to represent the wing by a superposition of source planes (representing one integration) the pressure distribution for which has then to be integrated over the wing area to give the drag force (representing two further integrations). By suitable choice of oblique axes, one of these integrations (representing the drag of one elementary source plane on a spanwise line across the wing) may be conducted without knowledge of the wing ordinates. The resulting expression for the drag is then, by suitable modification, expressed as a double integral, the kernel of which includes the expression for the wing slope

$$Z'(k) = \frac{2c}{t}\,\frac{dz(x)}{dx}\ , \quad k = \frac{x}{c} \tag{1}$$

where c is the root chord, t the root maximum thickness, and z(x) is the wing semi-ordinate at the root at a distance x aft of the wing apex. The remainder of the kernel is a complicated expression involving the parameters

$$\left.\begin{aligned} \sigma &= \frac{1}{4}\,A \tan \Lambda_o = \frac{1}{4}\,A \tan \Lambda_{\frac{1}{2}} + \frac{1}{2} \\ \text{and}\quad \alpha &= \frac{1}{4}\sqrt{M^2-1}\ \cot \Lambda_o \end{aligned}\right\} \tag{2}$$

which may best be expressed in different ways according to the position of the Mach lines from the apex on the wing.

In the following paragraphs these various expressions are written down, and are extracted from the final results enunciated in the Appendix I. In some particular cases these expressions are capable of considerable simplification,

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and these cases are noted, and the simplifications conducted, in the final section of this paragraph.

2.1 <u>Both Leading and Trailing Edges Subsonic</u> (<u>CASE I</u>)

This is Case I of the Appendix I, and the relevant result is obtained from equations (31) and (32) of that Appendix. We have from these that

$$\left.\begin{aligned}\frac{D}{qt^2\cot^2\Lambda_o} &= \frac{2\sigma^2}{\pi}\int_o^1 d\ell\int_o^\ell Z'(k)Z'(\ell)\Big[P(\lambda,\kappa)+P(\kappa,\lambda)\Big]dk\\ &= \frac{\sigma^2}{\pi}\int_o^1\int_b^1 Z'(k)Z'(\ell)\Big[P(\lambda,\kappa)+P(\kappa,\lambda)\Big]dk\,d\ell\end{aligned}\right\}\qquad(3)$$

where $P(a,b)$ is a function of α, as well as of a and b, which is given by

$$\begin{aligned}P(a,b) = {}&\frac{1}{a^2-\alpha^2}\left[1-\frac{b}{\sqrt{b^2-\alpha^2}}-\frac{a}{(a+b)^2}\left(2a-\sqrt{b^2-\alpha^2}\right)\right]+\frac{2ab}{(a+b)^2(a-b)\sqrt{b^2-\alpha^2}}\\ &+\frac{a}{(a+b)^2\sqrt{a^2-\alpha^2}}\left[\frac{4b}{a+b}+\frac{b^2+2ab-a^2}{a^2-\alpha^2}\right]\left[\cosh^{-1}\left(\frac{ab-\alpha^2}{\alpha|a-b|}\right)-\cosh^{-1}\left(\frac{a}{\alpha}\right)\right]\\ &-\frac{2b}{(a+b)^2\sqrt{b^2-\alpha^2}}\left(\frac{b^2}{b^2-\alpha^2}-\frac{2a}{a+b}\right)\ell n\left(\frac{b}{\alpha}\right)\end{aligned}\qquad(4)$$

and where

$$\left.\begin{aligned}\lambda &= \sigma-\ell\\ \text{and }\kappa &= \sigma-k\end{aligned}\right\}\qquad(5)$$

The parameters α and σ have been defined in equation (2), and $Z'(k)$ has been defined in equation (1). It should be noted in the present case that λ and κ are both greater than α over the range of integration.

2.2 <u>Both Leading and Trailing Edges Supersonic</u> (<u>CASE II</u>)

This is Case II of Appendix I, and the relevant result is obtained from equations (33) and (34). Thus we find that

$$\frac{D}{qt^2\cot^2\Lambda_o} = \sigma^2\int_o^1\frac{\left[Z'(\ell)\right]^2}{\sqrt{\alpha^2-\lambda^2}}\,d\ell+\frac{2\sigma^2}{\pi}\int_o^1 d\ell\int_o^\ell Z'(k)Z'(\ell)Q(\lambda,\kappa)\,dk\qquad(6)$$

/where...

where

$$Q(\lambda,\kappa) = \frac{4\lambda\kappa}{(\lambda+\kappa)^3}\left[\frac{1}{\sqrt{\alpha^2-\lambda^2}}\cos^{-1}\left(\frac{-\lambda}{\alpha}\right) - \frac{1}{\sqrt{\alpha^2-\kappa^2}}\cos^{-1}\left(\frac{\kappa}{\alpha}\right)\right]$$

$$+ \frac{\kappa^2+2\kappa\lambda-\lambda^2}{(\alpha^2-\lambda^2)(\kappa+\lambda)^2}\left[\frac{\lambda}{\sqrt{\alpha^2-\lambda^2}}\cos^{-1}\left(\frac{\lambda}{\alpha}\right)-1\right]$$

$$+ \frac{\lambda^2+2\lambda\kappa-\kappa^2}{(\alpha^2-\kappa^2)(\kappa+\lambda)^2}\left[\frac{-\kappa}{\sqrt{\alpha^2-\lambda^2}}\cos^{-1}\left(\frac{\kappa}{\alpha}\right)-1\right] \tag{7}$$

is a function of α as well as of λ and κ; and where as before

$\lambda = \sigma - \ell$,

and $\kappa = \sigma - k$.

Both λ and κ are in this case less than α.

2.3 Leading Edge Subsonic, and Trailing Edge Supersonic (CASE III)

This is Case III of the Appendix I, and the relevant result is obtained from equations (35) and (36). It is the most complicated case to deal with, since we must divide the range of integration over the wing surface so as to isolate the regions of 'subsonic' and 'supersonic' flow. We have from the Appendix I that

$$\frac{D}{qt^2\cot^2\Lambda_0} = \sigma^2\int_{\sigma-\alpha}^{1}\frac{[Z'(\ell)]^2}{\sqrt{\alpha^2-\lambda^2}}d\ell + \frac{2\sigma^2}{\pi}\int_0^{\sigma-\alpha}d\ell\int_0^{\ell}Z'(\ell)Z'(k)\left[P(\lambda,\kappa)+P(\kappa,\lambda)\right]dk$$

$$+ \frac{2\sigma^2}{\pi}\int_{\sigma-\alpha}^{1}Z'(\ell)\left[\int_{\sigma-\alpha}^{\ell}Z'(k)Q(\lambda,\kappa)dk + \int_0^{\sigma-\alpha}Z'(k)R(\lambda,\kappa)dk\right]d\ell \tag{8}$$

Here the functions P and Q, and the variables λ and κ have been defined before in equations (4), (5) and (7) above; the function:

$R(\lambda,\kappa)$...

$$
\begin{aligned}
R(\lambda,\kappa) = {} & \frac{1}{(\lambda+\kappa)\sqrt{\kappa^2-\alpha^2}}\left[\frac{\kappa(\kappa+\lambda)}{\alpha^2-\lambda^2}+\frac{\lambda}{\lambda+\kappa}\,\frac{\kappa^2-\alpha^2}{\alpha^2-\lambda^2}-\frac{2\kappa\lambda}{\kappa^2-\lambda^2}\right] \\
& + \frac{\lambda}{(\lambda+\kappa)^2\sqrt{\alpha^2-\lambda^2}}\left[\frac{4\kappa}{\lambda+\kappa}-\frac{\kappa^2+2\lambda\kappa-\lambda^2}{\alpha^2-\lambda^2}\right]\cos^{-1}\left[\frac{\lambda\kappa-\alpha^2}{\alpha(\kappa-\lambda)}\right] \\
& + \frac{2\kappa}{(\lambda+\kappa)^2\sqrt{\kappa^2-\alpha^2}}\left(\frac{2\lambda}{\lambda+\kappa}-\frac{\kappa^2}{\kappa^2-\alpha^2}\right)\ln\left(\frac{\kappa}{\alpha}\right) \\
& - \left[\frac{\kappa^2+2\kappa\lambda-\lambda^2}{(\alpha^2-\lambda^2)(\kappa+\lambda)^2}+\frac{\kappa^2-2\kappa\lambda-\lambda^2}{(\kappa^2-\alpha^2)(\lambda+\kappa)^2}\right] \\
& - \frac{\lambda}{(\lambda+\kappa)^2\sqrt{\alpha^2-\lambda^2}}\left[\frac{4\kappa}{\lambda+\kappa}-\frac{\kappa^2+2\lambda\kappa-\lambda^2}{\alpha^2-\lambda^2}\right]\cos^{-1}\left(\frac{\lambda}{\alpha}\right) \\
& + \frac{\kappa}{(\lambda+\kappa)^2\sqrt{\kappa^2-\alpha^2}}\left[\frac{4\lambda}{\lambda+\kappa}+\frac{\lambda^2+2\lambda\kappa-\kappa^2}{\kappa^2-\alpha^2}\right]\cosh^{-1}\left(\frac{\kappa}{\alpha}\right) \qquad (9)
\end{aligned}
$$

2.4 Some Particular Conditions.

These general formulae are rather too complicated for integration in terms of known functions, unless (perhaps) the surface shape were expressable in a particularly simple manner. However for a few particular types of planform, or ranges of Mach Number, it is possible to develope expressions which permit integration even for complicated section shapes. These further cases, which are arrived at by assigning certain limiting values to the parameters α and σ, will now be distinguished.

Case IV : $\sigma \to \infty$, $\alpha = \mu_o\sigma$, $(\mu_o < 1)$.

This is a particular example of Case I; from (4), expanding $P(\lambda,\kappa)$ as a series of powers in $\frac{1}{\sigma}$:

$$
P(\lambda,\kappa) = C_1+C_2\,\mathrm{sgn}(\kappa-\lambda) + \frac{2-\mu_o^2}{2\sigma^2(1-\mu_o^2)^{3/2}}\ln\left|\frac{1}{\kappa-\lambda}\right| + \frac{1}{2\sigma^2(1-\mu_o^2)^{3/2}}\,\frac{(2-\mu_o^2)\sigma-\kappa}{\lambda-\kappa} + O\left(\frac{1}{\sigma^3}\right)
$$

Noting that, since $Z(0) = Z(1) = 0$,

$$
\int_o^1 d\ell\int_o^\ell Z'(k)Z'(\ell)dk = \int_o^1 Z'(\ell)Z(\ell)d\ell = 0
$$

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we have in the equations (3) after substituting from (10) that:

$$
\left.\begin{aligned}
\frac{D}{qt^2\cot^2\Lambda_o} &= \frac{2}{\pi}\,\frac{2-\mu_o^2}{(1-\mu_o^2)^{3/2}}\int_0^1 d\ell\int_0^\ell Z'(k)Z'(\ell)\,\ln\left|\frac{1}{k-\ell}\right|dk + O\left(\frac{1}{\sigma}\right)\\
&= \frac{1}{\pi}\,\frac{2-\mu_o^2}{(1-\mu_o^2)^{3/2}}\int_0^1\int_0^1 Z'(k)Z'(\ell)\ln\left|\frac{1}{k-\ell}\right|dk\,d\ell + O\left(\frac{1}{\sigma}\right)
\end{aligned}\right\} \quad (11)
$$

<u>Case V</u> : <u>$\alpha \to \infty$, $\sigma = O(1) > 1$.</u>

This is a particular example of Case II; and from (7) expanding $Q(\lambda,\kappa)$ as a series of $\frac{1}{\alpha}$, we easily establish that $Q(\lambda,\kappa) = O\left(\frac{1}{\alpha^3}\right)$. Thus in equation (8) since

$$
\left(\cot^2\Lambda_o\right)\, t^2\sigma^2/\alpha = \frac{1}{\beta}\left(\frac{t}{c}\right)^2 sc = \frac{1}{\beta}\left(\frac{t}{c}\right)^2 S
$$

where S is the wing plan area, it follows that

$$
\frac{\beta D}{q\left(\frac{t}{c}\right)^2 S} = \int_0^1 \left[Z'(\ell)\right]^2 d\ell + O\left(\frac{1}{\alpha^2}\right) \quad (12)
$$

<u>Case VI</u> : <u>$\alpha \to 0$, $\sigma > 1$.</u>

This is a particular example of Case I, again; expanding $P(\lambda,\kappa)$ about $\alpha = 0$, we find from (4) that

$$
P(\lambda,\kappa) = \frac{2(\kappa-\lambda)}{(\lambda+\kappa)^3}\ln\alpha + \frac{1}{\lambda^2}\ln\kappa - \frac{1}{(\lambda+\kappa)^2}\left[\frac{\kappa}{\lambda} + \frac{2\kappa}{\kappa-\lambda}\right] + \left[\frac{1}{\lambda^2} + \frac{2(\kappa-\lambda)}{(\lambda+\kappa)^3}\right]\ln\left(\frac{1}{\kappa-\lambda}\right) + O(\alpha^2)
$$

so that

$$
P(\lambda,\kappa) + P(\kappa,\lambda) = \frac{1}{\lambda^2}\ln\kappa + \frac{1}{\kappa^2}\ln\lambda + \left(\frac{1}{\lambda^2} + \frac{1}{\kappa^2}\right)\ln\left|\frac{1}{\kappa-\lambda}\right| - \frac{1}{\lambda\kappa} + O(\alpha^2) \quad (13)
$$

Thus in (3)

$$
\frac{D}{qt^2\cot^2\Lambda_o} = \frac{2\sigma^2}{\pi}\int_0^1 d\ell\int_0^\ell Z'(k)Z'(\ell)\left[\frac{1}{\lambda^2}\ln\left|\frac{\kappa}{\kappa-\lambda}\right| + \frac{1}{\kappa^2}\ln\left|\frac{\lambda}{\lambda-\kappa}\right| - \frac{1}{\lambda\kappa}\right]dk
$$

We note that as $\sigma \to \infty$, it follows from equation (11) that the error is not merely $O(\alpha^2)$ but $O(\mu_o^2) = O\left(\frac{\alpha^2}{\sigma^2}\right)$. Also

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if σ is finite and > 1, we may put $O(\alpha^2) = O\left(\frac{\alpha^2}{\sigma^2}\right)$; thus in general we have that

$$\frac{D}{qt^2\cot^2\Lambda_o} = \frac{2\sigma^2}{\pi}\int_0^1 d\ell\int_0^\ell Z'(k)Z'(\ell)\left[\frac{1}{\lambda^2}\ln\left|\frac{\kappa}{\kappa-\lambda}\right| + \frac{1}{\kappa^2}\ln\left|\frac{\lambda}{\lambda-\kappa}\right| - \frac{1}{\lambda\kappa}\right]dk + O(\mu_o^2)$$

$$= \frac{\sigma^2}{\pi}\int_0^1\int_0^1 Z'(k)Z'(\ell)\left[\frac{1}{\lambda^2}\ln\left|\frac{\kappa}{\kappa-\lambda}\right| + \frac{1}{\kappa^2}\ln\left|\frac{\lambda}{\lambda-\kappa}\right| - \frac{1}{\lambda\kappa}\right]dk\,d\ell + O(\mu_o^2)$$

$$= \frac{\sigma^2}{\pi}\left[\int_0^1\int_0^1 Z'(k)Z'(\ell)\frac{1}{\lambda^2}\ln\frac{\kappa^2}{(\kappa-\lambda)^2}dk\,d\ell - \left(\int_0^1 \frac{Z'(k)}{\kappa}dk\right)^2\right] + O(\mu_o^2) \qquad (14)$$

It is sometimes more convenient to refer to the half chord sweepback ($\Lambda_{\frac{1}{2}}$) rather than that of the leading edge (Λ_o); in general

$$\cot\Lambda_a = \cot\Lambda_o\left(1-\frac{a}{\sigma}\right) \qquad (15)$$

so that in (14) we obtain an expression for $D/qt^2\cot^2\Lambda_{\frac{1}{2}}$ by replacing σ by $(\sigma-\frac{1}{2})$.

3. The Case of the Section with Discontinuous Surface Slope.

It is unnecessary to examine in detail each stage of the arguments which lead eventually to the equations (3) to (14) if the expression $Z'(k)$ is not continuous as has been assumed. For the most part, it will be sufficient to show that for the kind of discontinuities envisaged the integrals involved in the equations remain convergent.

The integrals are all, in fact, absolutely convergent if $Z'(k)$ is merely bounded over the region of integration[x], so that the only case in doubt is that in which $Z'(k)$ is somewhere singular. The most commonly occurring case is that for which $Z'(k)$ has a singularity of half-order at the

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[x]Except in the case where $\sigma = 1$, $a = 0$, which corresponds to the delta wing at $M = 1$. Here the double integrals are non-convergent unless $Z'(k)$ is zero at $k = 1$ (i.e. the trailing edge is cusped).

leading-edge : this corresponds to a wing section with a bluff, or radiused, nose such as is common to all conventional low speed aerofoils. It is easy to show that for such a condition the integrals remain convergent, unless the line of the singularity on the wing plan form is 'supersonic', (i.e. it lies ahead of the Mach lines); in this case the single integral in equations (6) or (8) is non-convergent.

One further point does arise in connection with the evaluation of the drag of such bluff nosed sections : R.T.Jones has pointed out[5] that the drag of such sections has to be estimated with care as there is, on the basis of the linearised theory, an edge force at the nose which is not included by the usual methods of integrating normal pressures over the wing to yield the resultant drag. This edge force exactly cancels a term which occurs in the integrations and which has the form

$$\int_0^1 dk \int_0^1 \frac{Z'(k)Z'(\ell)}{k - \ell} d\ell.$$

This repeated integral is non-zero if, and only if, the function $Z'(k)$ is singular in the range of integration; and the edge force exists if, and only if, the slope (given by the same function) is infinite at the nose. It will be noted that in our arguments in Appendix I where $Z'(k)$ is assumed continuous, and so bounded, a repeated integral of the form of that given above has been isolated and shown to vanish; this occurs in the derivation of equation (25) based on the arguments advanced in Appendix II. If we are to consider singularities in $Z'(k)$ it follows that this repeated integral no longer vanishes : however, as we show in Appendix III, it yields a value which is precisely cancelled by the inclusion of the appropriate edge force as obtained from ref.5, so that its neglect is quite justified whether or not the surface slope is anywhere infinite.

Of course, the usual reservations must be made if we are to consider the drag of round-nosed sections by the linear theory : but in so far as it has yet to be shown whether the answer obtained by it is at all reasonable, it is evidently desirable to have a method of estimating it.

4. The Particular Cases for which the Expressions for the Drag are Simplified.

In the discussion of para.2, we found that in the three

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conditions designated as Cases IV, V and VI, the expression for D was greatly simplified. We shall consider these one by one.

Case IV is that of the infinite swept wing with subsonic edges, or of the highly swept wing of finite aspect ratio. That is, it is the result for the condition

$$\sigma \to \infty \text{ i.e. } \frac{s}{c \cot \Lambda_o} \to \infty, \text{ and } \mu_o = \beta \cot \Lambda_o < 1.$$

which may be interpreted either as

$$\frac{s}{c} \to \infty \ (\cot \Lambda_o \text{ finite}), \ \beta \cot \Lambda_o < 1.$$

$$\text{or } \cot \Lambda_o \to 0 \ (\tfrac{s}{c} \text{ not small}), \ \beta < \frac{1}{\cot \Lambda_o}$$

Because it holds for the infinite wing, we should expect the result to agree with that already published for such wings. A comparison with the result of ref.2 reveals this agreement with one exception - that a factor $(3-\mu_o^2)$ is replaced in (11) by the factor $(2-\mu_o^2)$.

This discrepancy is at first sight irreconcilable: however it will be noted that the results for ref.2 were derived by considering the drag of the infinite wing as the limit of that for an untapered wing of large span, whereas here it is the limit of that for a fully tapered wing. The fact emerges that the limiting values are different, which may be substantiated by examining the trends in numerical computations carried out for both types of wing (tapered and untapered) for plan-forms of finite aspect ratio. Figure 1 shows some results extracted from ref.1, where it will be seen that for the same root wing section, Mach Number, and half-chord-line sweepback, the drag of the fully tapered wing is considerably smaller at high aspect-ratio than that of the untapered wing. The fact that they are not the same is understandable, but that they do not tend to approach each other as the aspect ratio increases is at first sight a surprising result, for the wing plan forms in the limit would seem to coincide near the root, where most of the drag is developed.

Yet the span of the tapered wing is only a half that of the untapered wing with which it is compared, and its

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area is only a quarter of that of the untapered wing, and of course this applies even as the aspect ratio increases to infinity. For this reason the smaller drag of the tapered wing does not seem unreasonable. There are, too, other properties of wings of infinite span which suggest that the value of the property depends on the manner in which one reaches the limiting answer. Consider, for example, the aerodynamic centre of a laminar delta wing: this is (theoretically) at the $\frac{2}{3}$ c. point of the root-chord independent of aspect ratio, however large this may be: yet as we allow the aspect ratio of a straight (rectangular) wing to increase, on the other hand, its aerodynamic centre moves aft to the $\frac{1}{2}$-chord point of the root-chord. In the limit both delta wing and rectangular wing appear to have the same planform if the aspect ratio is infinite, and yet the aerodynamic centre of the wings is at a different position.

Returning to our consideration of Case IV, since, apart from the slight modification mentioned - which only affects the variation of the drag with μ_o, - the results of this case have already been discussed in ref.2, we shall not draw any further conclusions here about the effect of section shape on the drag of any of the planforms the case includes.

Nor is Case V of any great interest : this result (equation (12)) merely states that if α is sufficiently large, to a good approximation the drag coefficient of the wing (based on its plan area) is the same as that for the same wing section in two-dimensional flow at the free-stream Mach Number. The conditions for this to be true are that

$$\alpha = \frac{\beta s}{c} \to \infty, \quad \sigma = \frac{s}{c \cot \Lambda_o} = O(1) > 1.$$

which implies either that the Mach Number of flight is very high, or that the aspect ratio is high but the sweepback is small: in either case the flow is two-dimensional outside the Mach Cone from the wing apex, and this latter affects only a negligible portion of the wing. Strictly the two conditions may be expressed as :

either $\beta \to \infty$, $\left(\frac{s}{c} \text{ and } \cot \Lambda_o \text{ not small}\right)$

or $\frac{s}{c} \to \infty$, $\cot \Lambda_o = O(\frac{s}{c})$, ($\beta$ not small).

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This deduction about the drag coefficient of such planforms has been noticed before, of course.

Of greatest interest to us here is Case VI, where

$$\alpha \rightarrow 0 \text{ and } \sigma > 1.$$

As is shown in the discussion of the formulae, it is sufficient in deriving these results if we assume

$$\mu_o \rightarrow 0, \text{ and } \sigma > 1.$$

These conditions may be interpreted as meaning that either the Mach Number is near unity, or that the leading-edge sweepback is high (provided in either case the trailing-edge is swept back.) The consideration of the limit $M \rightarrow 1$ is not strictly justifiable on the basis of the linear theory, so that we shall restrict our interpretation to the condition that the wing sweep is high so that

$$\cot \Lambda_o \rightarrow 0. \quad (\beta \text{ finite})$$

Whereas in Case IV, we could also consider such highly swept wings, but only those of finite aspect ratio, we may now consider also those of small aspect ratio as well. For this reason we shall distinguish Case IV as that of the 'slender wing', since it conforms with the same kind of restrictions as the 'slender-body' theory applied to lifting wing surfaces, developed by R.T.Jones[6].

5. The Drag of Slender Wings.

5.1 Limits of Applicability.

Equation (14) gives us the expression for the drag of slender wings, if we interpret these as satisfying the restrictions noticed above. Our first problem is to ascertain within what limits practical wing shapes are likely to have approximately the same drag as that found by this slender-wing theory. The drag of fully-tapered wings has only been extensively computed for those with double-wedge sections, and we shall therefore base our deductions on the results for this wing-section.

For the double-wedge section with a maximum thickness

/at...

at a fraction n of the chord, it is shown in Appendix IV that

$$\frac{D}{qt^2\cot^2\Lambda_o} = \frac{2\sigma^2}{\pi}\left\{\frac{\ln(\sigma-n)}{\sigma(\sigma-1)n(1-n)} - \frac{\ln[\sigma(1-n)]}{(\sigma-n)(\sigma-1)n} - \frac{\ln[(\sigma-1)n]}{\sigma(\sigma-n)(1-n)}\right\} \quad (16)$$

This expression has been evaluated for $n = 0.2(0.1)0.6$, and the results are shown in figure 2, where from (15),

$$\frac{D}{qt^2\cot^2\Lambda_{\frac{1}{2}}} = \left(1-\frac{1}{2\sigma}\right)^2\frac{D}{qt^2\cot^2\Lambda_o}$$

is shown plotted against

$$(1/A\tan\Lambda_{\frac{1}{2}}) = 1/(4\sigma-2).$$

It will be seen that over the greater part of the range of aspect ratio (for $A\tan\Lambda_{\frac{1}{2}} < 7$), the forward position of the maximum thickness reduces the drag, as opposed to the conditions existing for very high aspect-ratio wings.

The variation in drag with

$$\sqrt{M^2-1}\,\cot\Lambda_{\frac{1}{2}} = \mu_o\left(\frac{2\sigma}{1-2\sigma}\right)$$

for wings of double-wedge section (with maximum thickness at 0.5 chord) is shown in figure 3, where the results are taken from ref. 1 with some slight modification near (but not at) $\sqrt{M^2-1}\,\cot\Lambda_{\frac{1}{2}} = 0$ to include the fact that $D \propto \mu_o^2$ near $\mu_o = 0$*. It will be seen that provided $A\tan\Lambda_{\frac{1}{2}} > 2.5$, the change in drag is under 10% for values of $\sqrt{M^2-1}\,\cot\Lambda_{\frac{1}{2}}$ lying between 0 and 0.3, so that the slender-wing approximation $(\mu_o \to 0)$, is valid over a reasonable range of Mach Number. This does not apply to the results of lower aspect ratio.

Moreover the relative drag of the various sections does not vary greatly with Mach Number. This is shown in figure 4, where the relative drag is shown for various maximum thickness positions as a function of $\sqrt{M^2-1}\,\cot\Lambda_{\frac{1}{2}}$,

/for...

*In other respects the agreement of the present calculations with those of ref.6, for M = 1, is excellent.

for a wing having $A\tan\Lambda_{\frac{1}{2}} = 6$. These results are again obtained from ref.1. What change there is appears to emphasise the reduction in drag by putting the maximum thickness line forward - which is, physically, what one would expect.

5.2 The Drag of Polygonal Sections.

The drag of polygonal sections may be evaluated quite easily using the method of Appendix IV. For instance, using the results of the Appendix, it may be shown that if

$$Z'(k) = n_\nu, \text{ for } k_{\nu-1} < k < k_\nu$$

for all $\nu = 1, 2, \ldots p$, where $k_o = 0$, and $k_p = 1$; then

$$\frac{D}{qt^2\cot^2\Lambda_o} = \frac{2\sigma^2}{\pi}\sum_{\nu=1}^{p}\sum_{\mu=0}^{\nu-1}(n_\nu - n_{\nu+1})(n_\mu - n_{\mu+1})\, S\,(k_\nu, \ell_\mu) \tag{17}$$

where $S(\ell, k) = S(k, \ell) = \dfrac{(\ell - k)^2}{(\sigma-\ell)(\sigma-k)}\ln|k-\ell| + \left(\dfrac{k-\ell}{\sigma-\ell}\right)\ln(\sigma-k) + \left(\dfrac{\ell-k}{\sigma-k}\right)\ln(\sigma-\ell)$

and n_o and n_{p+1} are taken as zero.

This formula is quoted forreference only and no numerical examples have been worked : the drag of such sections may of course quite readily be estimated not only for the condition $\mu_o = 0$, but for other Mach Numbers, using standard methods.

5.3 Drag of Wing Sections Expressed as a Fourier Series.

If we suppose that the ordinates of the wing-section are expressed as a Fourier Series, i.e.

$$Z(k) = \sum_{n=1}^{\infty} a_n \sin n\theta, \text{ where } k = \frac{1+\cos\theta}{2} \tag{18}$$

the double integral of (14) is capable of evaluation in terms of the coefficients a_n: this is done in Appendix V.

Some of the simpler results may be noted here : for the ellipse, $a_n = 1$, $a_2 = a_3 = \ldots = 0$. From Appendix V, we may then calculate that,

$/ \dfrac{D}{qt^2\cot^2\Lambda_{\frac{1}{2}}} \ldots$

$$\left.\begin{aligned}&\frac{D}{qt^2\cot^2\Lambda_{\frac{1}{2}}} = \pi\coth^2\xi\ , \text{ where } \cosh\xi = 2\sigma-1\\ &\text{or } \frac{D}{qt^2\cot^2\Lambda_{\frac{1}{2}}} = \frac{\pi}{4}\,\frac{(2\sigma-1)^2}{\sigma(\sigma-1)}\end{aligned}\right\} \qquad (19)$$

The Joukowski section is given by

$$Z(k) = \frac{16}{3\sqrt{3}}\left[k^{\frac{1}{2}}(1-k)^{3/2}\right]$$

i.e. $a_1 = \frac{4}{3\sqrt{3}}$, $a_2 = -\frac{2}{3\sqrt{3}}$, $a_3 = a_4 = \ldots = 0$; we then calculate that

$$\frac{D}{qt^2\cot^2\Lambda_{\frac{1}{2}}} = \frac{8\pi}{27}\left(\frac{1+e^{-2\xi}}{1+e^{-\xi}}\right)^2\left(3 - e^{-2\xi}\right) \qquad (20)$$

where ξ is defined in the previous equation.

The 'double-cusped' aerofoil section is given by

$$Z(k) = 8\left[k(1-k)\right]^{3/2}$$

i.e. $a_1 = \frac{3}{4}$, $a_2 = 0$, $a_3 = -\frac{1}{4}$, $a_4 = a_5 = \ldots = 0$; then

$$\frac{D}{qt^2\cot^2\Lambda_{\frac{1}{2}}} = \frac{3\pi}{8}\ (1+e^{-2\xi})^2\,(2-e^{-2\xi}) \qquad (21)$$

5.4 Drag of Sections expressed as Polynomials

The drag of sections whose shape is given by polynomial expressions involves in its evaluation a great deal of labour, and only the simplest examples are quoted.

For the biconvex section, where

$$Z(k) = 4k(1-k)$$

evaluation of the integral (14) reveals that

$$\frac{D}{qt^2\cot^2\Lambda_{\frac{1}{2}}} = \frac{8}{\pi}(2\sigma-1)^2\left[(2\sigma-1)\ln\left(\frac{\sigma}{\sigma-1}\right) - 1 - \sigma(\sigma-1)\left(\ln\frac{\sigma}{\sigma-1}\right)^2\right] \qquad (22)$$

In this integration, the answer is obtained in terms of

/Euler's

Euler's dilogarithmic integral $L_2(x)$. However, these terms may be expressed in terms of logarithms, if note is made of the identities

$$L_2(x) + L_2(1-x) = L_2(1) - \ln|x| \ln|1-x| .$$

$$L_2\left(\frac{1}{x}\right) + L_2\left(\frac{1}{1-x}\right) = -\tfrac{1}{2}\left[\ln\left|\frac{x-1}{x}\right|\right]^2 .$$

These relations are also used in estimating the drag of a section such as that given by

$$Z(k) = \frac{3\sqrt{3}}{2} k^{\frac{1}{2}}(1-k)$$

which has the 'conventional' bluff leading edge and angled trailing edge; calculation reveals that

$$\frac{D}{qt^2\cot^2\Lambda_{\frac{1}{2}}} = \frac{27}{16\pi}(2\sigma-1)^2\left[\left(\frac{3\sigma+1}{\sigma}\right)\sigma^{\frac{1}{2}}\coth^{-1}\sigma^{\frac{1}{2}} - 3\right]\left[1 - \left(\frac{\sigma-1}{\sigma}\right)\sigma^{\frac{1}{2}}\coth^{-1}\sigma^{\frac{1}{2}}\right] \tag{23}$$

5.5 Drag of Reversed Sections.

The well-known reversed flow theorem applies to the drag of the slender wings*: the drag is the same if the flow direction is reversed. This may easily be verified from equation (14b) if the value of $(D/qt^2\cot^2\Lambda_{\frac{1}{2}})$ is written down, and $(1-k)$, $(1-\ell)$, and $(1-\sigma)$ are substituted for k, ℓ and σ respectively: the resulting double integral is then identical in both cases. The physical interpretation of this result is that the drag of a sweptforward wing is the same as that of the reversed (sweptback) wing with a reversed section shape. As there appears little interest in the drag of sweptforward wings at the moment, no data on the drag of such wings with sections asymmetrical fore-and-aft is presented here. For wing-sections with fore-and-aft

/symmetry...

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*It is not established, yet, that the reversed flow theorem may be extended to include the cases where there are edge forces present (as in the application to the bluff-nosed wing). The fact that the present work indicates that the reversed flow theorem does apply in this particular case, lends substance to the belief that such an extension may be possible.

symmetry, the drag is the same, of course, for the reversed (sweptforward) wing.

5.6 Discussion of the Results.

The variation of the drag with aspect ratio and sweepback, of the five sections given by equations (19) - (23), is illustrated in figure 5. Once again it will be seen that a reduction of the aspect ratio tends to reduce the drag of those wings which have a forward position of the maximum thickness, as is the case with double-wedge wings. Moreover (particularly as the aspect ratio is reduced so that the trailing-edge becomes nearly unswept) we see that the sections with cusped trailing-edges appear to have least drag.

The mathematical explanation of this latter effect lies in the fact that the pressure distribution near the trailing-edge becomes singular as $A\tan\Lambda_{\frac{1}{2}} \to 2$. Thus, for slender wings with a finite trailing-edge angle there is a logarithmic singularity in drag at $A\tan\Lambda_{\frac{1}{2}} = 2$ (and of course $\sqrt{M^2-1}\cot\Lambda_{\frac{1}{2}} = 0$) whilst the drag remains finite for sections with a cusped edge; a bluff edge, as on the elliptic section, produces a simple singularity at $A\tan\Lambda_{\frac{1}{2}} = 2$. However it is doubtful whether this has much physical meaning: the theoretically high drag of delta-like wings near $M = 1$ has not been recorded experimentally.

In figure 6 the drag of the various sections investigated is shown in relation to that of the double-wedge wing (with maximum thickness at half-chord). The basis of this comparison is that the root maximum thickness and half-chord-line sweep is the same for both sections, and the relative drag is shown as varying with $A\tan\Lambda_{\frac{1}{2}}$: the comparison serves further to emphasise the remarks made previously about the effect of maximum thickness position and the trailing-edge shape.

A particularly apt way of accounting for the effect of the maximum thickness position, is to quote the comparison as above except that not only the maximum thickness but also the sweepback of the maximum thickness line (instead of the half-chord line) is the same for both sections. This is done in figure 7, where the relative drag on this basis is

/plotted...

plotted against $A \tan \Lambda_{\frac{1}{2}}$ as before : however, we now note that since $\Lambda_{\frac{1}{2}}$ is not the same for both wings, the aspect ratio A will also differ. The effect of this artifice is to eliminate to a large extent the variation of the relative drag with the parameter $A \tan \Lambda_{\frac{1}{2}}$. The only broad tendency to be observed from figure 7 is that for sections with convex curvature aft of the maximum thickness there is an increase in drag as $A \tan \Lambda_{\frac{1}{2}}$ falls, whilst the opposite is true if there is a concave curvature : the effect is more pronounced if the maximum thickness is back at the half-chord - that is, if the curvature near the rear of the section is also more pronounced. For sections without any curvature (i.e. the double wedge sections), the relative drag, on this basis, is not greatly affected by change in aspect-ratio.

6. Some Comments on Future Work.

It will be evident that even in the case of the slender wing, and where the section shape may be expressed in a closed form, considerable effort is required to obtain an expression for the drag. There seems no doubt that for other wing shapes or Mach Number conditions, where the variation of α must also be taken into account (in which case the kernel of the double integral is far more complicated), and where the wing section (if described even by an approximating closed form of expression) is more involved, the evaluation and computation of the double integral for the drag presents a tedious problem.

Bearing this in mind, some work has been done to find the best method of performing a numerical integration of the drag double-integral. This in itself presents a problem as proper account has to be taken of the singularities in the integrand : if the wing edges are subsonic, which is the case of main interest, the range of integration is over the unit rectangle, with a logarithmic singularity across a diagonal ($k = \ell$) and (if the wing is bluff nosed) with a singularity of half-order along each of the two adjacent sides. However, the formula for numerical integration, once set down, is applicable whether or not an expression for the wing shape is known in a closed form, and the value of the kernel of the double integral (independent of the expressions for the surface slope) may quite easily be assimilated within the weighting factors of the formulae.

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This method of computation therefore promises to be easy to apply, once the fundamental calculations are complete.

The method has something in common with Beane's approach to the problem of the drag of a tapered biconvex wing[4] by using an approximating polygon to the section, but is likely to be easier to apply, and to be a more accurate and more flexible method.

As the problems of the method are peculiar to the subject of Numerical Integration, it has been thought desirable to discuss these in a separate report, which it is hoped will shortly be published.

References

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4.	B. Beane	The Characteristics of Wings having Biconvex Sections. J.Ae.Sci., Vol. 18, No. 1. Jan. 1951. pp. 7 - 19.
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Appendix I

Mathematical Formulation of the Expression for the Drag.

A uniform plane distribution of sources in the plane $z = 0$, bounded upstream by the lines $\frac{y}{m} = \pm (x-\xi)$ for $x \geqslant \xi$, will produce (in the plane of the sources) a pressure distribution[2] given by

$$\frac{\pi\beta}{2\varepsilon}\frac{\Delta p}{q} = \Re\left\{\frac{\mu}{\sqrt{1-\mu^2}}\left\{\cosh^{-1}\left[\frac{(x-\xi)-\mu^2(y/m)}{\mu|x-\xi-(y/m)|}\right] + \cosh^{-1}\left[\frac{(x-\xi)+\mu^2(y/m)}{\mu|x-\xi+\frac{y}{m}|}\right]\right\}\right\}H(j)$$

where $H(j) = 1$ if $j > 0$ and $H(j) = 0$ if $j < 0$, and where

$$\left.\begin{aligned} j &= (x-\xi) - \beta|y| \text{ if } \mu < 1 \\ &= (x-\xi) - (|y|/m) \text{ if } \mu > 1 \end{aligned}\right\} \qquad (1)$$

This expression may be written more conveniently as

$$\left.\begin{aligned} \frac{\pi\beta}{2\varepsilon}\frac{\Delta p}{q} &= \Re\left\{\frac{\mu}{\sqrt{1-\mu^2}}\cosh^{-1}\left[\frac{(2-\mu^2)(x-\xi)^2-\beta^2y^2}{|\mu^2(x-\xi)^2-\beta^2y^2|}\right]\right\}H(j) \\ &= g(x-\xi,|y|,m), \text{ say.} \end{aligned}\right\} \qquad (2)$$

The quantity ε will be termed the source plane strength, and is, in fact, related to the angular deflection of the flow normal to the source plane : we have that

$$\lim_{z\to 0} w = (\operatorname{sgn} z)U\varepsilon\, H[m(x-\xi)-|y|] \qquad (3)$$

Consider now the following superposition of source-planes :

(i) a source-plane bounded by $y = \pm m(0)x$ of strength $z'(0)$

(ii) a distribution of source-planes bounded by $y = \pm m(\xi)(x-\xi)$ of infinitesimal strength $z''(\xi)d\xi$, for all $0 \leqslant \xi \leqslant c$.

(iii) a source-plane bounded by $y = \pm m(c)(x-c)$ of strength $z'(c)$.

Further, we suppose that, if $s > 0$,

$$\frac{1}{m(\xi)} = \frac{1}{m(0)} - \frac{\xi}{s} > 0 \qquad (4)$$

Then from (3), using (4), we find that for this field of flow :

/(sgn z)...

$$\begin{aligned}(\operatorname{sgn} z)\frac{w}{U}\bigg|_{z=0} &= z'(0)H\Big[m(0)x-|y|\Big]+\int_0^c z''(\xi)H\Big[m(\xi)(x-\xi)-|y|\Big]\,d\xi \\ &\quad -z'(c)H\Big[m(c)(x-c)-|y|\Big] \\ &= z'(0)H\left[x-\frac{|y|}{m(0)}\right]+\int_0^c z''(\xi)\left[\left(x-\frac{|y|}{m(0)}\right)-\xi\left(1-\frac{|y|}{s}\right)\right]d\xi \\ &\quad -z'(c)H\left[\left(x-\frac{|y|}{m(0)}\right)-c\left(1-\frac{|y|}{s}\right)\right]\end{aligned}$$

Performing the integration, we find that, for $|y| \leq s$

$$\left.\begin{aligned} w\Big|_{z=0} &= 0 \quad \text{if} \quad x < \frac{|y|}{m(0)} \quad \text{or } x > \frac{|y|}{m(c)} + c \\ &= (\operatorname{sgn} z)U\, z'\left[\left(x-\frac{|y|}{m(0)}\right)\Big/\left(1-\frac{|y|}{s}\right)\right] \text{ if } \frac{|y|}{m(0)} < x < \frac{|y|}{m(c)} + c \end{aligned}\right\} \tag{5}$$

Hence the superposition of source-planes described above satisfies the boundary conditions for an arrowhead wing of semi-span s, with root chord c, and with swept-back leading and trailing edges. The leading edge is the lines $|y|=m(0)x$, and the wing section is geometrically similar at all spanwise positions, the surface at the root being given by $|z|=z(x)$, and the surface slope being everywhere continuous.

From (2), the pressure distribution on this wing is given by

$$\begin{aligned}\frac{\pi\beta}{2}\frac{\Delta p}{q} &= z'(0)g\Big[x,|y|,m(0)\Big] + \int_0^c z''(\xi)g\Big[x-\xi,|y|,m(\xi)\Big]\,d\xi \\ &\quad -z'(c)g\Big[x-c,|y|,m(c)\Big]\end{aligned} \tag{6}$$

The drag force on the wing is found to be

$$D = 4q\int_0^s dy\int_{\frac{y}{m(0)}}^{\frac{y}{m(c)}+c}\left\{\left(\frac{\Delta p}{q}\right)z'\left[\left(x-\frac{y}{m(0)}\right)\Big/\left(1-\frac{y}{s}\right)\right]\right\}dx \tag{7}$$

For convenience we shall now change the notation and variables, as detailed below. We put

$$\left.\begin{aligned}&\alpha = \beta s/c, \quad \sigma = s/cm(0), \quad Z'(a) = \left(\frac{t}{2c}\right)z'\left(\frac{a}{c}\right) \\ &\frac{x}{c} = \sigma\eta + (1-\eta)\ell \quad \frac{y}{s} = \eta\,, \quad \frac{\xi}{c} = k\end{aligned}\right\} \tag{8}$$

/where

where t is chosen as the maximum wing thickness (at the root).

After some calculation we find that

$$\left.\begin{aligned} &\beta m(\xi) = \frac{\alpha}{\sigma - k} \text{ and } \sigma > 1, \text{ from (4)} \\ &\text{and } \frac{\left(x - \frac{y}{m(0)}\right)}{\left(1 - \frac{y}{s}\right)} = \ell c, \text{ in (7).} \end{aligned}\right\} \tag{9}$$

Using the notation of equations (8) and (9), we find from (7) that

$$\frac{D}{q} = 2\, st \int_0^1 \int_0^1 \left(\frac{\Delta p}{q}\right) Z'(\ell)(1-\eta)\, d\eta\, d\ell \tag{10}$$

Again, from (6), in the new notation,

$$\left.\begin{aligned} \frac{\pi\beta}{2}\frac{\Delta p}{q} = \frac{\alpha}{2}\left(\frac{t}{c}\right)\Big\{Z'(0)h(\lambda,\sigma,\eta) &+ \int_0^1 Z''(k)h(\lambda,\kappa,\eta)\,dk \\ &- Z'(1)h(\lambda,\sigma-1,\eta)\Big\} \end{aligned}\right. \tag{11}$$

where, from equations (2), (8) and (9)

$$\left.\begin{aligned} &Z''(k) = \frac{dZ'(k)}{dk} \\ &\lambda = \sigma - \ell \\ &\kappa = \sigma - k \\ &h(\lambda,\kappa,\eta) = \Re\left\{\frac{1}{\sqrt{\kappa^2-\alpha^2}}(\cosh^{-1}X)H(j')\right\} \\ &X = \left\{\left(\frac{2\kappa^2}{\alpha^2}-1\right)\left[\kappa-(1-\eta)\lambda\right]^2 - \eta^2\kappa^2\right\} \Big/ \left|\left[\kappa-(1-\eta)\lambda\right]^2 - \eta^2\kappa^2\right| \\ &\text{and } j' = (\kappa-\lambda) + (\lambda-\alpha)\eta \quad \text{if } \kappa > \alpha \\ &\phantom{\text{and } j'} = (\kappa-\lambda) \quad \text{if } \kappa < \alpha. \end{aligned}\right\} \tag{12}$$

Substituting (11) in (10), and taking the integration with respect to η first

$$\begin{aligned} \frac{D}{q} = \left(\frac{2}{\pi}\frac{s^2}{c^2}\right)t^2 \int_0^1 d\ell \int_0^1 Z'(\ell)(1-\eta)\Big\{Z'(0)h(\lambda,\sigma,\eta) &+ \int_0^1 Z''(k)h(\lambda,\kappa,\eta)\,dk \\ &- Z'(1)h(\lambda,\sigma-1,\eta)\Big\}\, d\eta. \end{aligned} \tag{13}$$

Consider the repeated integral:

$/ \int_0^1 \ldots$

$$\int_0^1 d\eta \int_0^1 Z''(k)(1-\eta)h(\lambda,\kappa,\eta)dk.$$

The function $Z''(k)(1-\eta)h(\lambda,\kappa,\eta)$ is continuous within the range of integration except along certain lines, k = constant, and the integral of it with respect to k between $k = 0$ and 1 is convergent for all η between 0 and 1, provided that $Z'(k)$ is nowhere singular (which is a necessary condition for the applicability of the linear theory). Thus we may interchange the order of integration, and (13) may be written as

$$\frac{D}{q\left[m(0)\right]^2 t^2} = \frac{2\sigma^2}{\pi}\int_0^1 \Big[Z'(0)I(\lambda,\sigma) + \int_0^1 Z''(k)I(\lambda,\kappa)dk - Z'(1)I(\lambda,\sigma-1)\Big] Z'(\ell)d\ell \qquad (14)$$

where

$$I(\lambda,\kappa) = \int_0^1 (1-\eta)h(\lambda,\kappa,\eta)d\eta. \qquad (15)$$

If we put

$$J(\eta) = \int \frac{\cosh^{-1}X}{\sqrt{\kappa^2-\alpha^2}}(1-\eta)d\eta \qquad (16)$$

then, from (12) in (15), we have that

$$\left.\begin{aligned} I(\lambda,\kappa) &= \mathcal{R}\left\{J(1) - J(0)\right\} \quad \text{if} \quad \lambda < \kappa \\ &= \mathcal{R}\left\{J(1) - J\left(\frac{\lambda-\kappa}{\lambda-\alpha}\right)\right\} \quad \text{if} \quad \lambda > \kappa > \alpha \\ &= 0 \quad \text{if} \quad \lambda > \kappa,\ \kappa < \alpha. \end{aligned}\right\} \qquad (17)$$

But, in (16), integrating by parts

$$J(\eta) = -\tfrac{1}{2}(1-\eta)^2 \frac{\cosh^{-1}X}{\sqrt{\kappa^2-\alpha^2}} + \frac{1}{2\sqrt{\kappa^2-\alpha^2}}\int \frac{(1-\eta)^2}{\sqrt{X^2-1}}\frac{\partial X}{\partial \eta}d\eta.$$

In this expression we may calculate that, from (12),

$$\frac{\partial X}{\partial \eta} = \frac{4(\kappa^2-\alpha^2)\kappa^2\eta\left[\kappa-(1-\eta)\lambda\right]}{\alpha^2(1-\eta)\left[(\kappa-\lambda)+(\kappa+\lambda)\eta\right]\left|\left[\kappa-(1-\eta)\lambda\right]^2-\eta^2\kappa^2\right|}$$

and also,

$$\sqrt{X^2-1} = \frac{2\kappa}{\alpha^2}\sqrt{\kappa^2-\alpha^2}\sqrt{\left\{\left[\kappa-(1-\eta)\lambda\right]^2-\alpha^2\eta^2\right\}}\left|\kappa-(1-\eta)\lambda\right| \Big/ \left|\left[\kappa-(1-\eta)\lambda\right]^2-\eta^2\kappa^2\right|.$$

Thus, since $\left[\kappa-(1-\eta)\lambda\right] > 0$ for the range of values of $\eta,\kappa,$

/and...

and λ in which we are interested (see equation (17)), substituting these relations it follows that

$$J(\eta) = -\frac{(1-\eta)^2\cosh^{-1}X}{2\sqrt{\kappa^2-\alpha^2}} + \kappa\int\frac{\eta(1-\eta)}{\left[(\kappa-\lambda)+(\kappa+\lambda)\eta\right]}\frac{d\eta}{\Psi^{\frac{1}{2}}}$$

$$\text{where } \Psi = (\lambda^2-\alpha^2)\eta^2 + 2\lambda(\kappa-\lambda)\eta + (\kappa-\lambda)^2$$

$$= a\eta^2 + 2b\eta + c \text{ say.}$$

To effect an integration we first resolve into partial fractions

$$\frac{\eta(1-\eta)}{\left[(\kappa-\lambda)+(\kappa+\lambda)\eta\right]} = -\frac{\eta}{\kappa+\lambda} + \frac{2\kappa}{(\lambda+\kappa)^2} + \frac{2\kappa(\lambda-\kappa)}{(\lambda+\kappa)^2\left[(\kappa-\lambda)+(\kappa+\lambda)\eta\right]}$$

$$= A\eta + B + \frac{C}{\eta-\eta_o}, \text{ say, where } \eta_o = \frac{\lambda-\kappa}{\lambda+\kappa}$$

Upon integration we then have that,

$$J(\eta) = -\frac{(1-\eta)^2\cosh^{-1}X}{2\sqrt{\kappa^2-\alpha^2}} + \kappa\left\{\frac{A}{a}\Psi^{\frac{1}{2}} + \left(B-\frac{b}{a}A\right)\frac{1}{a^{\frac{1}{2}}}\cosh^{-1}\left(\frac{a\eta+b}{\sqrt{b^2-ac}}\right)\right.$$

$$\left. -\frac{C}{\kappa+\lambda}\left(\frac{1}{\Psi^{\frac{1}{2}}}\right)\Bigg|_{\eta=\eta_o}\cosh^{-1}\left[\frac{(a\eta_o+b)\eta+(b\eta_o+c)}{(\eta-\eta_o)\sqrt{b^2-ac}}\right]\right\}$$

Substituting for $A, B, C, a, b, c, X, \Psi$, and η_o we then find that

$$J(\eta) = -\frac{(1-\eta)^2}{2\sqrt{\kappa^2-\alpha^2}}\cosh^{-1}\left\{\frac{\left(\frac{2\kappa^2}{\alpha^2}-1\right)\left[\kappa-(1-\eta)\lambda\right]^2 - \eta^2\kappa^2}{\left|\left[\kappa-(1-\eta)\lambda\right]^2-\eta^2\kappa^2\right|}\right\}$$

$$-\frac{\kappa}{(\kappa+\lambda)(\lambda^2-\alpha^2)}\sqrt{\left\{\left[\kappa-(1-\eta)\lambda\right]^2-\alpha^2\eta^2\right\}}$$

$$+\frac{\kappa}{(\lambda+\kappa)\sqrt{\lambda^2-\alpha^2}}\left[\frac{2\kappa}{(\lambda+\kappa)}-\frac{\lambda(\lambda-\kappa)}{\lambda^2-\alpha^2}\right]\cosh^{-1}\left[\frac{(\lambda^2-\alpha^2)\eta+\lambda(\kappa-\lambda)}{\alpha|\kappa-\lambda|}\right]$$

$$+\frac{2\kappa^2\,\mathrm{sgn}(\kappa-\lambda)}{(\lambda+\kappa)^2\sqrt{\kappa^2-\alpha^2}}\cosh^{-1}\left\{\mathrm{sgn}(\kappa-\lambda)\frac{\left[(\alpha^2+\lambda\kappa)\eta+\kappa(\kappa-\lambda)\right]}{\alpha\left[(\kappa-\lambda)+(\kappa+\lambda)\eta\right]}\right\} \tag{18}$$

In particular, in (17), we require the values of

$$J(1) = -\frac{\kappa\sqrt{\kappa^2-\alpha^2}}{(\kappa+\lambda)(\lambda^2-\alpha^2)} + \frac{\kappa}{(\lambda+\kappa)\sqrt{\lambda^2-\alpha^2}}\left[\frac{2\kappa}{\lambda+\kappa}+\frac{\lambda(\kappa-\lambda)}{\lambda^2-\alpha^2}\right]\cosh^{-1}\left[\frac{\lambda\kappa-\alpha^2}{\alpha|\kappa-\lambda|}\right]$$

$$+\frac{2\kappa^2\,\mathrm{sgn}(\kappa-\lambda)}{(\lambda+\kappa)^2\sqrt{\kappa^2-\alpha^2}}\cosh^{-1}\left[\mathrm{sgn}(\kappa-\lambda)\frac{\alpha^2+\kappa^2}{2\alpha\kappa}\right] \tag{19}$$

/J(0)...

$$J(0) = -\frac{\kappa(\kappa-\lambda)}{(\kappa+\lambda)(\lambda^2-\alpha^2)} + \frac{\kappa}{(\lambda+\kappa)\sqrt{\lambda^2-\alpha^2}}\left[\frac{2\kappa}{\lambda+\kappa} - \frac{\lambda(\lambda-\kappa)}{\lambda^2-\alpha^2}\right]\cosh^{-1}\frac{\lambda}{\alpha}$$

$$+\frac{1}{\sqrt{\kappa^2-\alpha^2}}\left[\frac{2\kappa^2}{(\lambda+\kappa)^2}-1\right]\cosh^{-1}\left(\frac{\kappa}{\alpha}\right) \text{for } \kappa > \lambda \tag{20}$$

and $\mathcal{R}\left\{J\left(\frac{\lambda-\kappa}{\lambda-\alpha}\right)\right\} = 0 \quad \text{for} \quad \lambda > \kappa > \alpha.$ (21)

Substitution of these expressions in (17) will yield the appropriate form of $I(\lambda,\kappa)$, which is in general discontinuous at $\kappa = \lambda$, where it has a logarithmic singularity, and at $\kappa = \alpha < \lambda$, where there is a simple discontinuity in its derivative with respect to κ. Let us now put

$$\left.\begin{aligned} &\frac{\partial}{\partial\kappa} I(\lambda,\kappa) = I_1(\lambda,\kappa) \text{ for } \kappa > \lambda \\ &\frac{\partial}{\partial\kappa} I(\lambda,\kappa) = I_2(\lambda,\kappa) \text{ for } \kappa < \lambda \ (\kappa \neq \alpha) \end{aligned}\right\} \tag{22}$$

Then both $I_1(\lambda,\kappa)$ and $I_2(\lambda,\kappa)$ have simple singularities at $\kappa=\lambda$ and a discontinuity at $\kappa = \alpha$. Using these definitions in (14), we have on integrating by parts with respect to κ, and taking the principle part of the improper integrals, that:

$$\frac{D}{q\left[m(0)\right]^2 t^2} = \frac{2\sigma^2}{\pi}\int_0^1 Z'(\xi) \lim_{\varepsilon\to 0}\left[Z'(\xi-\varepsilon)I(\lambda,\lambda+\varepsilon) - Z'(\xi+\varepsilon)I(\lambda,\lambda-\varepsilon)\right.$$

$$-Z'(\sigma-\alpha)I(\lambda,\alpha-\varepsilon)\Big|_{\lambda>\alpha} + Z'(\sigma-\alpha)I(\lambda,\alpha+\varepsilon)\Big|_{\lambda>\alpha} + \int_0^{\xi-\varepsilon} Z'(k)I_1(\lambda,\kappa)dk$$

$$\left.+\int_{\xi+\varepsilon}^1 Z'(k)I_2(\lambda,\kappa)dk\right]d\xi \tag{23}$$

Now, from (17), (19), and (20), if $\lambda > \alpha$, for sufficiently small ε,

$$I(\lambda,\lambda\pm\varepsilon) = -\frac{1}{2\sqrt{\lambda^2-\alpha^2}} + \frac{1}{2\sqrt{\lambda^2-\alpha^2}}\ln\left[\frac{2(\lambda^2-\alpha^2)}{\alpha\varepsilon}\right] - \frac{1}{2\sqrt{\lambda^2-\alpha^2}}\ln\left(\frac{\alpha}{\lambda}\right) + O(\varepsilon)$$

$$I(\lambda,\alpha+\varepsilon)\Big|_{\lambda>\alpha} = O(\varepsilon), \text{ and } I(\lambda,\alpha-\varepsilon)\Big|_{\lambda>\alpha} = 0.$$

If $\lambda < \alpha$, again for sufficiently small ε,

$$I(\lambda,\lambda+\varepsilon) = \frac{\pi}{2\sqrt{\alpha^2-\lambda^2}} + O(\varepsilon)$$

$$I(\lambda,\lambda-\varepsilon) = 0.$$

/Thus...

Thus (23) becomes

$$\frac{D}{q\left[m(0)\right]^2 t^2} = \sigma^2 \int_0^1 \frac{H(\alpha-\lambda)}{\sqrt{\alpha^2-\lambda^2}} \left[Z'(\ell)\right]^2 d\ell$$

$$+ \frac{2\sigma^2}{\pi} \int_0^1 Z'(\ell) \lim_{\varepsilon\to 0} \left[\int_0^{\ell-\varepsilon} Z'(k) I_1(\lambda,\kappa)\,dk + \int_{\ell+\varepsilon}^1 Z'(k) I_2(\lambda,\kappa)\,dk\right] \tag{24}$$

Excluding, for the moment, the range of α such that $\sigma \geqslant \sigma \geqslant \sigma-1$ (so that I_2 is discontinuous within the range of integration), the second (repeated) integral in this expression is capable of simplification. We note that at $\kappa = \lambda$, both $I_1(\lambda,\kappa)$ and $I_2(\lambda,\kappa)$ have the same singularity and that $Z'(k)$ is continuous, so let us put:

$$Z'(k) I_1(\lambda,\kappa) = \overset{*}{I}_1(\lambda,\kappa) + \frac{C(\lambda)}{\lambda-\kappa}\ ; \quad Z'(k) I_2(\lambda,\kappa) = \overset{*}{I}_2(\lambda,\kappa) + \frac{C(\lambda)}{\lambda-\kappa}$$

where $\overset{*}{I}_1$ and $\overset{*}{I}_2$ are now everywhere continuous except for a logarithmic singularity, and C is some function of λ which is also continuous. Then

$$\lim_{\varepsilon\to 0}\left(\int_0^{\ell-\varepsilon} \frac{C(\lambda)}{\lambda-\kappa}\,dk + \int_{\ell+\varepsilon}^1 \frac{C(\lambda)}{\lambda-\kappa}\,dk\right) = C(\lambda)\,\ell n\left(\frac{1-\ell}{\ell}\right).$$

Also we have that, on first changing the order of integration and then the symbols

$$\int_0^1 Z'(\ell)\left[\lim_{\varepsilon\to 0}\int_{\ell+\varepsilon}^1 \overset{*}{I}_2(\lambda,\kappa)\,dk\right] d\ell = \int_0^1 Z'(\ell)\left[\int_\ell^1 \overset{*}{I}_2(\lambda,\kappa)\,dk\right] d\ell$$

$$= \int_0^1 \left[\int_0^k Z'(\ell)\overset{*}{I}_2(\lambda,\kappa)\,d\ell\right] dk$$

$$= \int_0^1 \left[\int_0^\ell Z'(k)\overset{*}{I}_2(\kappa,\lambda)\,dk\right] d\ell.$$

Thus, substituting $\overset{*}{I}_1$ and $\overset{*}{I}_2$ for I_1 and I_2, using the above identity, and then returning to the original notation:

$$\int_0^1 Z'(\ell) \lim_{\varepsilon\to 0}\left[\int_0^{\ell-\varepsilon} Z'(k) I_1(\lambda,\kappa)\,dk + \int_{\ell+\varepsilon}^1 Z'(k) I_2(\lambda,\kappa)\,dk\right] d\ell$$

$$= \int_0^1 d\ell \int_0^\ell \left[Z'(\ell)\overset{*}{I}_1(\lambda,\kappa) + Z'(k)\overset{*}{I}_2(\kappa,\lambda)\right] dk + \int_0^1 Z'(\ell) C(\lambda)\,\ell n\left(\frac{1-\ell}{\ell}\right) d\ell$$

$$= \int_0^1 d\ell \int_0^\ell Z'(k) Z'(\ell)\left[I_1(\lambda,\kappa) + I_2(\kappa,\lambda)\right] dk + \left\{\int_0^1 d\ell \int_0^\ell \left[\frac{Z'(\ell)C(\lambda) - Z'(k)C(\kappa)}{\ell - k}\right] c \right.$$

$$\left. + \int_0^1 Z'(\ell) C(\lambda)\,\ell n\left(\frac{1-\ell}{\ell}\right) d\ell\right\} \tag{25}$$

/The...

The expression contained within the curly brackets in the last expression may be shown (see Appendix II) to be identically zero, so that on substituting from (25) in (24),

$$\frac{D}{q\left[m(0)\right]^2 t^2} = \sigma^2 \int_0^1 \frac{H(\alpha-\lambda)}{\sqrt{\alpha^2-\lambda^2}}\left[Z'(\ell)\right]^2 d\ell$$

$$+ \frac{2\sigma^2}{\pi}\int_0^1 d\ell \int_0^{\ell} Z'(\ell) Z'(k)\left[I_1(\lambda,\kappa) + I_2(\kappa,\lambda)\right] dk \qquad (26)$$

where the integrand of the second integral has now only a logarithmic singularity at $\kappa = \lambda$. To find the values of $I_1(\lambda,\kappa)$ and $I_2(\lambda,\kappa)$ we require the values of

$$\frac{\partial J(1)}{\partial \kappa} = -\frac{1}{(\lambda+\kappa)\sqrt{\kappa^2-\alpha^2}}\left[\frac{\kappa(\kappa+\lambda)}{\lambda^2-\alpha^2} + \frac{2\kappa^2}{\kappa^2-\lambda^2} + \frac{\lambda}{\lambda+\kappa}\,\frac{\kappa^2-\alpha^2}{\lambda^2-\alpha^2} - \frac{2\kappa}{\lambda+\kappa}\operatorname{sgn}(\kappa-\alpha)\right]$$

$$+ \frac{\lambda}{(\lambda+\kappa)^2\sqrt{\lambda^2-\alpha^2}}\left[\frac{4\kappa}{(\lambda+\kappa)} + \frac{\kappa^2+2\lambda\kappa-\lambda^2}{\lambda^2-\alpha^2}\right]\cosh^{-1}\frac{\lambda\kappa-\alpha^2}{\alpha\left|\kappa-\lambda\right|}$$

$$+ \frac{2\kappa\operatorname{sgn}(\kappa-\lambda)}{(\lambda+\kappa)^2\sqrt{\kappa^2-\alpha^2}}\left(\frac{2\lambda}{\lambda+\kappa} - \frac{\kappa^2}{\kappa^2-\alpha^2}\right)\cosh^{-1}\left[\operatorname{sgn}(\kappa-\lambda)\,\frac{\alpha^2+\kappa^2}{2\alpha\kappa}\right] \qquad (27)$$

and $$\frac{\partial J(0)}{\partial \kappa} = -\left[\frac{\kappa^2+2\kappa\lambda-\lambda^2}{(\lambda^2-\alpha^2)(\kappa+\lambda)^2} + \frac{\lambda^2+2\kappa\lambda-\kappa^2}{(\kappa^2-\alpha^2)(\lambda+\kappa)^2}\right]$$

$$+ \frac{\lambda}{(\lambda+\kappa)^2\sqrt{\lambda^2-\alpha^2}}\left[\frac{4\kappa}{\lambda+\kappa} + \frac{\kappa^2+2\lambda\kappa-\lambda^2}{\lambda^2-\alpha^2}\right]\cosh^{-1}\frac{\lambda}{\alpha}$$

$$+ \frac{\kappa}{(\lambda+\kappa)^2\sqrt{\kappa^2-\alpha^2}}\left[\frac{4\lambda}{\lambda+\kappa} + \frac{\lambda^2+2\lambda\kappa-\kappa^2}{(\kappa^2-\alpha^2)}\right]\cosh^{-1}\frac{\kappa}{\alpha} \qquad (28)$$

Then from equations (22), (17), and (21):

$$\left.\begin{aligned} I_1(\lambda,\kappa) &= \Re\left\{\frac{\partial J(1)}{\partial \kappa} - \frac{\partial J(0)}{\partial \kappa}\right\} \quad (\kappa > \lambda) \\ I_2(\lambda,\kappa) &= \Re\left\{\left[\frac{\partial J(1)}{\partial \kappa}\right] H(\kappa-\alpha)\right\} \quad (\kappa < \lambda) \end{aligned}\right\} \qquad (29)$$

Using (27) and (28) in (29) to find $I_1(\lambda,\kappa)$ and $I_2(\lambda,\kappa)$, and substituting for $I_1(\lambda,\kappa)$ and $I_2(\kappa,\lambda)$ in (26) we may then write down our general expression for the drag. If $\sigma \geqslant \alpha \gg \sigma-1$, then equation (24) must be used in place of (26). We may conveniently consider the precise form of this expression in certain particular cases, one by one.

/CASE I...

CASE I Both Wing Leading and Trailing Edges Subsonic. ($\alpha < \sigma - 1$)

In equation (29), we then have that $\kappa = \sigma - k \geqslant \sigma - 1 > \alpha$, and similarly $\lambda > \alpha$, so that

$$I_2(\lambda, \kappa) = \left\{ \frac{\partial J(1)}{\partial \kappa} \right\} .$$

We also note that $\partial J(0)/\partial \kappa$ may be expressed in the form

$$-\frac{\partial J(0)}{\partial \kappa} = I_3(\kappa, \lambda) + I_3(\lambda, \kappa) \qquad (30)$$

where from (28), we may infer that

$$I_3(\lambda, \kappa) = \frac{\kappa^2 + 2\kappa\lambda - \lambda^2}{(\lambda^2 - \alpha^2)(\kappa + \lambda)^2} - \frac{\lambda}{(\lambda + \kappa)^2 \sqrt{\lambda^2 - \alpha^2}} \left[\frac{4\kappa}{\lambda + \kappa} + \frac{\kappa^2 + 2\lambda\kappa - \lambda^2}{\lambda^2 - \alpha^2} \right] \cosh^{-1} \frac{\lambda}{\alpha} .$$

Thus in (26), putting $m(0) = \cot \Lambda_0$, we have that

$$\frac{D}{qt^2 \cot^2 \Lambda_0} = \frac{2\sigma^2}{\pi} \int_0^1 d\ell \int_0^{\ell} Z'(k) Z'(\ell) \left[P(\lambda, \kappa) + P(\kappa, \lambda) \right] dk \qquad (31a)$$

where

$$P(\lambda, \kappa) = \frac{\partial J(1)}{\partial \kappa} + I_3(\lambda, \kappa) . \qquad (32)$$

It follows immediately, since the integrand is symmetrical in k and ℓ that an alternative expression is

$$\frac{D}{qt^2 \cot^2 \Lambda_0} = \frac{\sigma^2}{\pi} \int_0^1 \int_0^1 Z'(k) Z'(\ell) \left[P(\lambda, \kappa) + P(\kappa, \lambda) \right] dk\, d\ell . \qquad (31b)$$

In (32), we may calculate from equations (27) and (30) that

$$P(\lambda, \kappa) = \left[\frac{1}{\lambda^2 - \alpha^2} - \frac{2\lambda^2}{(\lambda^2 - \alpha^2)(\lambda + \kappa)^2} - \frac{\kappa}{(\lambda^2 - \alpha^2)\sqrt{\kappa^2 - \alpha^2}} - \frac{2\kappa\lambda}{(\lambda + \kappa)^2 (\kappa - \lambda)\sqrt{\kappa^2 - \alpha^2}} \right.$$
$$\left. - \frac{\lambda \sqrt{\kappa^2 - \alpha^2}}{(\lambda + \kappa)^2 (\lambda^2 - \alpha^2)} \right] + \frac{\lambda}{(\lambda + \kappa)^2 \sqrt{\lambda^2 - \alpha^2}} \left[\frac{4\kappa}{\lambda + \kappa} + \frac{\kappa^2 + 2\kappa\lambda - \lambda^2}{\lambda^2 - \alpha^2} \right] \left[\cosh^{-1} \left(\frac{\lambda\kappa - \alpha^2}{\alpha |\kappa - \lambda|} \right) \right.$$
$$\left. - \cosh^{-1} \frac{\lambda}{\alpha} \right] - \frac{2\kappa}{(\lambda + \kappa)^2 \sqrt{\kappa^2 - \alpha^2}} \left(\frac{\kappa^2}{\kappa^2 - \alpha^2} - \frac{2\lambda}{\lambda + \kappa} \right) \ln(\kappa/\alpha) \qquad (32)$$

CASE II Leading and trailing edges supersonic. ($\alpha > \sigma$)

In (29), since $\kappa = \sigma - k < \alpha$ and similarly $\lambda < \alpha$, we have

/that...

that $I_2(\lambda,\kappa) = I_2(\kappa,\lambda) = 0$. Thus in (26), putting $I_1(\lambda,\kappa) = Q(\lambda,\kappa)$ say,

$$\frac{D}{qt^2\cot^2\Lambda_o} = \sigma^2\int_0^1 \frac{[Z'(\ell)]^2}{\sqrt{\alpha^2-\lambda^2}}d\ell + \frac{2\sigma^2}{\pi}\int_0^1 d\ell\int_0^{\ell} Z'(\ell)Z'(k)Q(\lambda,\kappa)dk. \qquad (33)$$

To find $Q(\lambda,\kappa)$ we must find the value of $I_1(\lambda,\kappa)$ for $\lambda < \kappa < \alpha$; in doing this we note that in (27),

$$\frac{\lambda\kappa-\alpha^2}{\alpha|\kappa-\lambda|} < -1, \text{ so that } I\left\{\cosh^{-1}\left(\frac{\lambda\kappa-\alpha^2}{\alpha|\kappa-\lambda|}\right)\right\} = \pi$$

and also

$$I\left\{\cosh^{-1}\left[\operatorname{sgn}(\kappa-\lambda)\left(\frac{\alpha^2+\kappa^2}{2\alpha\kappa}\right)\right]\right\} = 0.$$

Thus we may calculate from the equations (27) and (28) in (29) that

$$\begin{aligned} Q(\lambda,\kappa) = & \frac{4\lambda\kappa}{(\lambda+\kappa)^3}\left[\frac{1}{\sqrt{\alpha^2-\lambda^2}}\cos^{-1}\left(-\frac{\lambda}{\alpha}\right) - \frac{1}{\sqrt{\alpha^2-\kappa^2}}\cos^{-1}\left(\frac{\kappa}{\alpha}\right)\right] \\ & + \frac{\kappa^2+2\kappa\lambda-\lambda^2}{(\alpha^2-\lambda^2)(\kappa+\lambda)^2}\left[\frac{\lambda}{\sqrt{\alpha^2-\lambda^2}}\cos^{-1}\left(\frac{\lambda}{\alpha}\right)-1\right] \\ & + \frac{\lambda^2+2\lambda\kappa-\kappa^2}{(\alpha^2-\kappa^2)(\kappa+\lambda)^2}\left[\frac{\kappa}{\sqrt{\alpha^2-\kappa^2}}\cos^{-1}\left(\frac{\kappa}{\alpha}\right)-1\right] \end{aligned} \qquad (34)$$

<u>CASE III</u> <u>Leading edge subsonic, and trailing edge supersonic. ($\sigma-1 < \alpha < \sigma$)</u>

In this condition we use equation (24) in place of (26) and since $0 < \sigma-\alpha < 1$, so that $(\alpha-\lambda)$ is positive if $1 > \ell > \sigma-\alpha$, the first integral in this equation becomes

$$\int_0^1 \frac{H(\alpha-\lambda)}{\sqrt{\alpha^2-\lambda^2}}\left[Z'(\ell)\right]^2 d\ell = \int_{\sigma-\alpha}^1 \frac{[Z'(\ell)]^2}{\sqrt{\alpha^2-\lambda^2}}d\ell .$$

In the second (double) integral, we note that the integrand is (within the range of integration) continuous except for a discontinuity at $\kappa = \alpha$. It should be noted that the <u>individual</u> terms of the expressions for $I_1(\lambda,\kappa)$ and $I_2(\lambda,\kappa)$ have simple singularities at $\lambda = \alpha$ or $\kappa = \alpha$ though these singularities do not occur in I_1 or I_2 regarded as a whole.

Now for $\kappa < \lambda$ and also $\kappa < \alpha$, it follows from (29) that $I_2(\lambda,\kappa) = 0$ so that in (24)

/ $\int_0^1 \ldots$

$$\int_0^1 Z'(\ell)\lim_{\varepsilon\to 0}\left[\int_{\ell+\varepsilon}^1 Z'(k)I_2(\lambda,\kappa)dk\right]d\ell$$

$$= \int_0^{\sigma-\alpha} Z'(\ell)d\ell \lim_{\varepsilon\to 0}\int_{\ell+\varepsilon}^{\sigma-\alpha} Z'(k)I_2(\lambda,\kappa)dk$$

where $I_2(\lambda,\kappa)$ is now continuous over the range of integration. Hence, as in deriving (25) of this Appendix, changing first the order of integration and then the symbols, it follows on using the result of Appendix II that this expression may be shown to equal

$$\int_0^{\sigma-\alpha} Z'(\ell)d\ell\int_0^{\ell+\varepsilon} Z'(k)I_2(\kappa,\lambda)dk.$$

Thus in (24) the second integral becomes

$$\int_0^1 Z'(\ell)\lim_{\varepsilon\to 0}\left[\int_0^{\ell-\varepsilon} Z'(k)I_1(\lambda,\kappa)dk + \int_{\ell+\varepsilon}^1 Z'(k)I_2(\lambda,\kappa)dk\right]d\ell$$

$$= \int_0^{\sigma-\alpha} Z'(\ell)d\ell\int_0^{\ell} Z'(k)\left[I_1(\lambda,\kappa) + I_2(\kappa,\lambda)\right]dk$$

$$+ \int_{\sigma-\alpha}^1 Z'(\ell)d\ell\int_0^{\ell} Z'(k)I_1(\lambda,\kappa)dk$$

$$= \left\{\int_0^{\sigma-\alpha} Z'(\ell)d\ell\int_0^{\ell} Z'(k)\left[I_1(\lambda,\kappa) + I_2(\kappa,\lambda)\right]dk\right.$$

$$\left. + \int_{\sigma-\alpha}^1 Z'(\ell)d\ell\left(\int_0^{\sigma-\alpha} + \int_{\sigma-\alpha}^{\ell}\right)Z'(k)I_1(\lambda,\kappa)dk\right\}.$$

In the first integral over the region $0 \leq \ell \leq \sigma-\alpha$, $0 \leq k \leq \ell$, we have that $\kappa > \lambda > \alpha$, so that as in equation (4) above, we find that

$$I_1(\lambda,\kappa) + I_2(\kappa,\lambda) = P(\lambda,\kappa) + P(\kappa,\lambda).$$

Similarly in the region $\sigma-\alpha \leq \ell \leq 1$, $\sigma-\alpha \leq k \leq \ell$, we find that $\alpha > \kappa > \lambda$: thus as in (33) above, we may put

$$I_1(\lambda,\kappa) = Q(\lambda,\kappa).$$

Finally, in the intermediate region, where $\sigma-\alpha \leq \ell \leq 1$ and $0 \leq k \leq \sigma-\alpha$, we have $\lambda < \alpha < \kappa$; then we put

$$I_1(\lambda,\kappa) = R(\lambda,\kappa), \text{ say.}$$

Thus in (24) we have on summarizing all the above results:

/...

$$\frac{D}{qt^2\cot^2\Lambda_o} = \sigma^2 \int_{\sigma-\alpha}^{1} \frac{[Z'(\ell)]^2}{\sqrt{\alpha^2-\lambda^2}}\, d\ell$$

$$+ \frac{2\sigma^2}{\pi}\left\{\int_0^{\sigma-\alpha} Z'(\ell)\,d\ell \int_0^{\ell} Z'(k)\Big[P(\lambda,\kappa) + P(\kappa,\lambda)\Big]dk\right.$$

$$+\left[\int_{\sigma-\alpha}^{1} Z'(\ell)\,d\ell \int_{\sigma-\alpha}^{\ell} Z'(k)\,Q(\lambda,\kappa)\,dk\right.$$

$$\left.\left.+\int_{\sigma-\alpha}^{1} Z'(\ell)\,d\ell \int_0^{\sigma-\alpha} Z'(k)\,R(\lambda,\kappa)\,dk\right]\right\}. \tag{35}$$

Note that these expressions are not defined for $\alpha = \sigma-1$ or $\alpha = \sigma$, although the value of D for either of these conditions may be computed as the limiting cases, for $\alpha \to \sigma-1$ or σ. In equation (35) we have that $P(\lambda,\kappa)$ and $Q(\lambda,\kappa)$ are given by the equations (32) and (34) above, and $R(\lambda,\kappa)$ is the value of $I_1(\lambda,\kappa)$ for $\lambda < \alpha < \kappa$: i.e. from (27), (28) and (29)

$$R(\lambda,\kappa) = \frac{1}{(\lambda+\kappa)\sqrt{\kappa^2-\alpha^2}}\left[\frac{\kappa(\kappa+\lambda)}{\alpha^2-\lambda^2} - \frac{2\kappa\lambda}{\kappa^2-\lambda^2} + \frac{\lambda}{\lambda+\kappa}\,\frac{\kappa^2-\alpha^2}{\alpha^2-\lambda^2}\right]$$

$$+ \frac{\lambda}{(\lambda+\kappa)^2\sqrt{\alpha^2-\lambda^2}}\left[\frac{4}{\lambda+\kappa} - \frac{\kappa^2+2\lambda\kappa-\lambda^2}{\alpha^2-\lambda^2}\right]\cos^{-1}\frac{\lambda\kappa-\alpha^2}{\alpha(\kappa-\lambda)}$$

$$+ \frac{2\kappa}{(\lambda+\kappa)^2\sqrt{\kappa^2-\alpha^2}}\left(\frac{2\lambda}{\lambda+\kappa} - \frac{\kappa^2}{\kappa^2-\alpha^2}\right)\ln\left(\frac{\kappa}{\alpha}\right)$$

$$-\left[\frac{\kappa^2+2\kappa\lambda-\lambda^2}{(\alpha^2-\lambda^2)(\kappa+\lambda)^2} + \frac{\kappa^2-2\kappa\lambda-\lambda^2}{(\kappa^2-\alpha^2)(\lambda+\kappa)^2}\right]$$

$$- \frac{\lambda}{(\lambda+\kappa)^2\sqrt{\alpha^2-\lambda^2}}\left[\frac{4\kappa}{\lambda+\kappa} - \frac{\kappa^2+2\lambda\kappa-\lambda^2}{\alpha^2-\lambda^2}\right]\cos^{-1}\left(\frac{\lambda}{\alpha}\right)$$

$$+ \frac{\kappa}{(\lambda+\kappa)^2\sqrt{\kappa^2-\alpha^2}}\left[\frac{4\lambda}{\lambda+\kappa} + \frac{\lambda^2+2\lambda\kappa-\kappa^2}{\kappa^2-\alpha^2}\right]\cosh^{-1}\left(\frac{\kappa}{\alpha}\right) \tag{36}$$

Appendix II

Proof of an Integral Identity

We shall here prove the identity

$$\int_0^a dk \int_0^k \frac{f(\ell) - f(k)}{\ell - k}\, d\ell = -\int_0^a f(\ell)\,\ln\left(\frac{a-\ell}{\ell}\right) d\ell \tag{1}$$

Consider the integral

$$I(a) = \int_0^a f(\ell)\,\ln\left(\frac{a-\ell}{\ell}\right) d\ell\,. \tag{2}$$

/Since...

Since

$$\int_0^a \ln\left(\frac{a-\ell}{\ell}\right) d\ell = 0 \tag{3}$$

it follows that if $f(\ell)$ is continuous at $\ell = a$

$$I(a) = \int_0^a \left[f(\ell) - f(a)\right] \ln\left(\frac{a-\ell}{\ell}\right) d\ell \; . \tag{4}$$

Then differentiating

$$\frac{dI(a)}{da} = \int_0^a \left[\frac{f(\ell) - f(a)}{a-\ell} - f'(a) \ln\left(\frac{a-\ell}{\ell}\right)\right] d\ell \tag{5}$$

a sufficient condition to allow the differentiation under the sign of integration being that $f(\ell)$ is continuous at $\ell = a$, and integrable over the range $(0, a)$. From (3) in (5):

$$\frac{dI(a)}{da} = \int_0^a \left[\frac{f(\ell) - f(a)}{a - \ell}\right] d\ell$$

and so

$$I(a) - I(0) = \int_0^a \frac{dI(k)}{dk}\, dk = \int_0^a dk \int_0^k \frac{f(\ell) - f(k)}{k-\ell}\, d\ell \; . \tag{6}$$

But, using (3)

$$I(0) = \lim_{a\to 0} I(a) = \lim_{a\to 0} \int_0^a f(\ell) \ln\left(\frac{a-\ell}{\ell}\right) d\ell \tag{7}$$

If $f(\ell)$ is continuous at $\ell = 0$ it is easy then to show in (7) that $I(0) = 0$, so that from (6) and (1), the identity (1) is proved: subject to the sufficient conditions that $f(\ell)$ is continuous at $\ell = 0$ and a, and integrable over $(0, a)$.

Appendix III

The Edge Force

The argument which leads to the neglect of the terms in the curly brackets in equation (25) of Appendix I is given in Appendix II, where it is shown that this neglect is certainly justified if the function $f(\ell)$ of the Appendix II is continuous at the end points of the interval of integration (i.e. at $\ell = 0$ and a). Now in equation (25) of Appendix I we identify this function $f(\ell)$ with $Z'(\ell)C(\lambda)$, which from its definition and the use of equations (27), (28) and (29) of

/Appendix I...

Appendix I, may be shown to equal

$$\left[Z'(\ell)\right]^2 \Big/ 2\sqrt{\lambda^2-\alpha^2}.$$

This is, in general, continuous at the end points $\ell = 0$ and $\ell = 1$ of the range of integration, provided that the slope of the aerofoil section is continuous at the nose and the trailing edge. This is the assumption in Appendix I, and it is this fact which justifies the use of the results of Appendix II. However if the slope at either of these end points is infinite, then the arguments we have used must break down.

Let us suppose, for example, that $Z'(k) \sim \frac{a_1}{k^{\frac{1}{2}}}$ as $k \to 0$, which describes the usual type of singularity associated with a bluff nosed aerofoil. We shall attempt to deduce equation (26) of Appendix I from equation (24) for this case, where the previous arguments used no longer apply. Similar reasoning to the following may be used to discuss what happens if $Z'(k)$ is singular at $k = 1$ (i.e. if the wing section has a bluff trailing edge).

Suppose we subdivide the range of integration of the repeated integral in equation (24) of Appendix I:

$$\int_0^1 Z'(\ell) \lim_{\varepsilon \to 0} \left[\int_0^{\ell-\varepsilon} Z'(k) I_1(\lambda,\kappa) dk + \int_{\ell+\varepsilon}^1 Z'(k) I_2(\lambda,\kappa) dk \right] d\ell$$

$$= \lim_{\delta \to 0} \left\{ \int_\delta^1 Z'(\ell) \lim_{\varepsilon \to 0} \left[\int_\delta^{\ell-\varepsilon} Z'(k) I_1(\lambda,\kappa) dk + \int_{\ell+\varepsilon}^1 Z'(k) I_2(\lambda,\kappa) dk \right] d\ell \right\}$$

$$+ \lim_{\delta \to 0} \left\{ \int_0^\delta Z'(\ell) \left[\int_\delta^1 Z'(k) I_2(\lambda,\kappa) dk \right] d\ell + \int_\delta^1 Z'(\ell) \left[\int_0^\delta Z'(k) I_1(\lambda,\kappa) dk \right] d\ell \right\}$$

$$+ \lim_{\delta \to 0} \left\{ \int_0^\delta Z'(\ell) \lim_{\varepsilon \to 0} \left[\int_0^{\ell-\varepsilon} Z'(k) I_1(\lambda,\kappa) dk + \int_{\ell+\varepsilon}^\delta Z'(k) I_2(\lambda,\kappa) dk \right] d\ell \right\} \quad (1)$$

Using arguments similar to those employed in deducing equation (26) from (24) of Appendix I, we may show that

$$\lim_{\delta \to 0} \left\{ \int_\delta^1 Z'(\ell) \lim_{\varepsilon \to 0} \left[\int_\delta^{\ell-\varepsilon} Z'(k) I_1(\lambda,\kappa) dk + \int_{\ell+\varepsilon}^1 Z'(k) I_2(\lambda,\kappa) dk \right] d\ell \right\}$$

$$= \lim_{\delta \to 0} \left\{ \int_\delta^1 d\ell \int_\delta^\ell Z'(\ell) Z'(k) \left[I_1(\lambda,\kappa) + I_2(\kappa,\lambda) \right] dk \right\} \quad (2)$$

These arguments, including the results of Appendix II, are justifiable since $Z'(k)$ and $Z'(\ell)$ have no singularities in the range of integration.

/Now...

Now the second limit in the right-hand side of (1) may be shown to vanish as $\delta \to 0$; also the limit of the integral on the right hand side of (2) may be shown to exist, so that taking this limit and substituting in (1),

$$\int_0^1 Z'(\ell)\lim_{\varepsilon\to 0}\left[\int_0^{\ell-\varepsilon} Z'(k)I_1(\lambda,\kappa)dk + \int_{\ell+\varepsilon}^1 Z'(k)I_2(\lambda,\kappa)dk\right]d\ell$$

$$= \int_0^1 d\ell\int_0^{\ell} Z'(k)Z'(\ell)\left[I_1(\lambda,\kappa) + I_2(\kappa,\lambda)\right]dk + F \qquad (3)$$

where

$$F = \lim_{\delta\to 0}\left\{\int_0^{\delta} Z'(\ell)\lim_{\varepsilon\to 0}\left[\int_0^{\ell-\varepsilon} Z'(k)I_1(\lambda,\kappa)dk + \int_{\ell+\varepsilon}^{\delta} Z'(k)I_2(\lambda,\kappa)dk\right]d\ell\right\} \qquad (4)$$

But in this repeated integral (on the right hand side of (4)), the region of integration is confined to the immediate neighbourhood of the point $\ell = k = 0$. Thus it is permissible to replace $Z'(k)$ and $Z'(\ell)$ by $\frac{a}{k^{\frac{1}{2}}}$ and $\frac{a}{\ell^{\frac{1}{2}}}$ respectively; and further we may replace

$$I_1(\lambda,\kappa) \text{ by } \left\{c_1 - \frac{1}{2\sqrt{\sigma^2-\alpha^2}(\kappa-\lambda)} + \frac{2\sigma^2-\alpha^2}{2\sigma(\sigma^2-\alpha^2)^{3/2}}\ \ell n\left|\frac{1}{\kappa-\lambda}\right|\right\}$$

$$I_2(\lambda,\kappa) \text{ by } \left\{c_2 - \frac{1}{2\sqrt{\sigma^2-\alpha^2}(\kappa-\lambda)} + \frac{2\sigma^2-\alpha^2}{2\sigma(\sigma^2-\alpha^2)^{3/2}}\ \ell n\left|\frac{1}{\kappa-\lambda}\right|\right\}$$

where c_1 and c_2 are finite constants, as may be established on examination of equations (27) (28) and (29) of Appendix I. Thus we find that in (4),

$$\frac{2\sqrt{\sigma^2-\alpha^2}}{a^2}F = \lim_{\delta\to 0}\left\{\int_0^{\delta}\frac{1}{\ell^{\frac{1}{2}}}\lim_{\varepsilon\to 0}\left[\int_0^{\ell-\varepsilon} + \int_{\ell+\varepsilon}^{\delta}\frac{1}{k^{\frac{1}{2}}}\frac{1}{\ell-k}dk\right]d\ell\right\} = \lim_{\delta\to 0}\left\{\int_0^{\delta}\frac{2}{\ell}\coth^{-1}\sqrt{\frac{\delta}{\ell}}\,d\ell\right\}$$

$$\text{i.e. } F = -\frac{2a^2}{\sqrt{\sigma^2-\alpha^2}}\int_0^{\infty}\frac{1}{x}\coth^{-1}x\,dx = -\frac{a^2\pi^2}{4\sqrt{\sigma^2-\alpha^2}} \qquad (5)$$

Hence in equation (26) of Appendix I, the expression for the drag D remains unaltered, except that from (4) and (5) above there is an additional drag

$$\Delta D = -\frac{\pi}{2}q\left[m(0)\right]^2\sigma^2 a^2 t^2 \Big/ \sqrt{\sigma^2-\alpha^2} \qquad (6)$$

Suppose that the radius of curvature of the wing leading-edge at the centre-section is r: then we may calculate that

$$r = a^2 t^2 \big/ 2c \qquad (7)$$

so that in (6) we have, after substituting for σ and α, that

/ΔD...

$$\Delta D = -\frac{\pi q m(0) s r}{\sqrt{1-\beta^2 m^2(0)}} = -\frac{\pi q s \cos\Lambda_o r}{\sqrt{1-M^2\cos^2\Lambda_o}} \tag{8}$$

However, the corrected equation (25) of Appendix I is no longer the complete expression for the drag, since it omits the edge force acting on the wing, which is not included in the integrations of the pressure over the wing surface. This edge-force is given in reference 5 as a force normal to the leading edge of amount

$$\frac{\pi q \cos^2\Lambda_o r_n}{\sqrt{1-M^2\cos^2\Lambda_o}} \quad \text{per unit length}$$

where r_n is the nose radius measured normal to the leading edge. But from the geometry of the wing section,

$$r_n = \frac{a^2}{2}\left(\frac{t^2}{c}\right)\sec\Lambda_o\left(1-\frac{y}{s}\right) = r\sec\Lambda_o\left(1-\frac{y}{s}\right)$$

so that the contribution of this edge force to the drag is

$$\Delta D_e = 2\cos\Lambda_o\int_o^s\left(\frac{\pi q\cos^2\Lambda_o r_n}{\sqrt{1-M^2\cos^2\Lambda_o}}\right)\frac{dy}{\cos\Lambda_o}$$

$$= \frac{\pi q s\cos\Lambda_o r}{\sqrt{1-M^2\cos^2\Lambda_o}} \tag{9}$$

Comparing (9) with (8) we see that the contribution of this edge force exactly cancels the additional term arising from the integrations when the leading edge is rounded. Hence, we deduce that equation (25) of Appendix I, in the form given in that Appendix, is the correct expression for the total drag whether or not the leading edge of the wing section is rounded; if the leading edge is round, we have seen that the derivation of this equation leaves out a term which however is equal and opposite to that obtained from the edge force, which should also properly be included.

Appendix IV

Integration of the Expression for the Drag of a Double-Wedge Section

We have to evaluate the integral of equation (39) of the Appendix I:

$$\frac{D}{q\left[m(0)\right]^2 t^2} = \frac{\sigma^2}{\pi}\int_o^1\int_o^1 Z'(k)Z'(\ell)\left[\frac{2}{(\sigma-\ell)^2}\ell n\left|\frac{\sigma-k}{k-\ell}\right| - \frac{1}{(\sigma-k)(\sigma-\ell)}\right]dk\,d\ell \quad (1)$$

/for...

for the condition

$$
\left.\begin{aligned}
Z'(k) &= \frac{1}{n} \quad \text{for } 0 \leqslant k \leqslant n \\
&= -\frac{1}{1-n} \quad \text{for } n \leqslant k \leqslant 1
\end{aligned}\right\} \qquad (2)
$$

Now,

$$
\int \ln\left|\frac{\sigma-k}{k-\ell}\right| dk = \left[(k-\sigma)\ln(\sigma-k) + (\ell-k)\ln|k-\ell|\right]
$$

and

$$
\int\left(\int \ln\left|\frac{\sigma-k}{k-\ell}\right|\right)\frac{dk\,d\ell}{(\sigma-\ell)^2} = \left[\ln|k-\ell|\ln\left(\frac{\sigma-\ell}{\sigma-k}\right) + \left(\frac{\sigma-k}{\sigma-\ell}\right)\ln\left|\frac{k-\ell}{\sigma-k}\right| - \ln\left|\frac{k-\ell}{\sigma-\ell}\right|\right]
$$

$$
+ \int \ln\left(\frac{\sigma-\ell}{\sigma-k}\right)\frac{d\ell}{k-\ell}
$$

that is,

$$
\iint \ln\left|\frac{\sigma-k}{k-\ell}\right|\frac{dk\,d\ell}{(\sigma-\ell)^2} = \left[\ln|k-\ell|\ln\left(\frac{\sigma-\ell}{\sigma-k}\right) + \left(\frac{\ell-k}{\sigma-\ell}\right)\ln|k-\ell| - \left(\frac{\sigma-k}{\sigma-\ell}\right)\ln(\sigma-k) + \ln(\sigma-\ell)\right]
$$

$$
+ \iint\left[-\frac{1}{(k-\ell)(\sigma-k)} + \frac{1}{(k-\ell)^2}\ln\left(\frac{\sigma-k}{\sigma-\ell}\right)\right]dk\,d\ell \qquad (3)
$$

We note that

$$
\int_0^1\!\!\int_0^1 Z'(k)Z'(\ell)\left[\frac{2}{(k-\ell)(\sigma-k)} + \frac{2}{(k-\ell)^2}\ln\left(\frac{\sigma-k}{\sigma-\ell}\right) - \frac{1}{(\sigma-\ell)(\sigma-k)}\right]dk\,d\ell
$$

$$
= \int_0^1\!\!\int_0^1 Z'(k)Z'(\ell)\left[\frac{2\sigma-k-\ell}{(k-\ell)(\sigma-k)(\sigma-\ell)} - \frac{2}{(k-\ell)^2}\ln\left(\frac{\sigma-\ell}{\sigma-k}\right)\right]dk\,d\ell = 0 \qquad (4)
$$

as we may see by interchanging first the order of integration and then the symbols, which is justifiable since the integrand is regular over the range of integration.

Using equations (1) to (4), it follows that the drag may be expressed as

$$
\frac{D}{q[m(0)]^2 t^2} = \frac{2\sigma^2}{\pi}\sum_{p=1}^{3}\sum_{q=1}^{3} a_p a_q \mathcal{L}(k_p, \ell_q) \qquad (5)
$$

where $a_1 = -\frac{1}{n}$; $a_2 = \frac{1}{n(1-n)}$; $a_3 = -\frac{1}{1-n}$;

$k_1, \ell_1 = 0$; $k_2, \ell_2 = n$; $k_3, \ell_3 = 1$;

and $\mathcal{L}(k,\ell) = \ln|k-\ell|\ln\left(\frac{\sigma-\ell}{\sigma-k}\right) + \left(\frac{\ell-k}{\sigma-\ell}\right)\ln|k-\ell| - \left(\frac{\sigma-k}{\sigma-\ell}\right)\ln(\sigma-k) + \ln(\sigma-\ell)$.

/We . .

We note that

$$\mathcal{L}(k, k) = 0$$

and $$\mathcal{L}(k,\ell) + \mathcal{L}(\ell,k) = \frac{(\ell-k)^2}{(\sigma-\ell)(\sigma-k)} \ln|k-\ell| + \left(\frac{k-\ell}{\sigma-\ell}\right)\ln(\sigma-k) + \left(\frac{\ell-k}{\sigma-k}\right)\ln(\sigma-\ell)$$

$$= S(k,\ell), \text{ say.}$$

Thus in (5),

$$\frac{D}{q[m(0)]^2 t^2} = \frac{2\sigma^2}{\pi}\left[S(0,1) - \frac{1}{n} S(0,n) - \frac{1}{1-n} S(n,1)\right] \Big/ n(1-n)$$

$$= \frac{2\sigma^2}{\pi n(1-n)}\left[-\frac{1}{\sigma-1}\ln\sigma + \frac{1}{\sigma}\ln(\sigma-1) - \frac{n}{\sigma(\sigma-n)}\ln n + \frac{1}{\sigma-n}\ln\sigma - \frac{1}{\sigma}\ln(\sigma-n)\right.$$

$$\left. - \frac{(1-n)}{(\sigma-1)(\sigma-n)}\ln(1-n) + \frac{1}{\sigma-1}\ln(\sigma-n) - \frac{1}{\sigma-n}\ln(\sigma-1)\right]$$

$$= \frac{2\sigma^2}{\pi}\left[\frac{\ln(\sigma-n)}{\sigma(\sigma-1)n(1-n)} - \frac{\ln[\sigma(1-n)]}{(\sigma-n)(\sigma-1)n} - \frac{\ln[(\sigma-1)n]}{\sigma(\sigma-n)(1-n)}\right]$$

Appendix V

Integration of the Expression for the Drag of a Wing Section expressed as a Fourier Series.

We have to evaluate the integral of equation (39) of Appendix I,

$$\frac{D}{qt^2\cot^2\Lambda_{\frac{1}{2}}} = \left(1-\frac{1}{2\sigma}\right)^2 \frac{\sigma^2}{\pi}\left[2\int_0^1 \frac{Z'(\ell)}{(\sigma-\ell)^2}\int_0^1 Z'(k)\ln\left|\frac{\sigma-k}{k-\ell}\right| dk\, d\ell - \left(\int_0^1 \frac{Z'(k)}{(\sigma-k)}dk\right)^2\right] \tag{1}$$

where

$$Z(k) = \sum_{n=1}^{\infty} a_n \sin n\theta, \text{ if } k = \frac{1+\cos\theta}{2} \tag{2}$$

and similarly,

$$Z(\ell) = \sum_{m=1}^{\infty} a_n \sin m\phi, \text{ if } \ell = \frac{1+\cos\phi}{2} \tag{3}$$

and for convenience, we put $\sigma = \dfrac{1+\cosh\xi}{2} \quad (\xi > 0)$. (4)

To procede with the integration, we note the value of the following definite integrals:

$$\int_0^\pi \frac{\sin n\theta\sin\theta}{\cosh\xi-\cos\theta}\, d\theta = \pi\exp(-n\xi) \quad (n > 0, \xi > 0) \tag{5}$$

/P..

$$P\int_0^\pi \frac{\sin n\theta\ \sin\theta}{\cos\phi-\cos\theta}\,d\theta = \pi\cos n\phi \quad (n>0) \tag{6}$$

$$\int_0^\pi \frac{\cos n\theta}{\cosh\xi-\cos\theta}\,d\theta = \frac{\pi\exp(-|n|\xi)}{\sinh\xi} \quad (\xi>0) \tag{7}$$

$$\int_0^\pi \frac{\cos n\theta}{(\cosh\xi-\cos\theta)^2}\,d\theta = \frac{\pi\exp(-|n|\xi)}{\sinh^3\xi}\left[|n|\sinh\xi+\cosh\xi\right]$$

$$(\xi>0) \tag{8}$$

Now, after an integration by parts, since $Z(0) = Z(1) = 0$, and using equations (2) and (4)

$$\int_0^1 Z'(k)\ln\left|\frac{\sigma-k}{k-\ell}\right|dk = P\int_0^1 Z(k)\left[\frac{1}{\sigma-k}-\frac{1}{\ell-k}\right]dk$$

$$= P\int_0^\pi \sum_{n=1}^\infty a_n\sin n\theta\sin\theta\left[\frac{1}{\cosh\xi-\cos\theta}-\frac{1}{\cos\phi-\cos\theta}\right]d\theta$$

Supposing that term by term integration is permissible, from (5) and (6),

$$\int_0^1 Z'(k)\ln\left|\frac{\sigma-k}{k-\ell}\right|dk = \pi\sum_{n=1}^\infty a_n\left[\exp(-n\xi)-\cos n\phi\right] \tag{9}$$

Again, from (3) and (9), integrating term by term

$$\int_0^1\frac{Z'(\ell)}{(\sigma-\ell)^2}\left[\int_0^1 Z'(k)\ln\left|\frac{\sigma-k}{k-\ell}\right|dk\right]d\ell = -4\pi\sum_{m=1}^\infty\sum_{n=1}^\infty m a_m a_n\int_0^\pi\frac{(e^{-n\xi}-\cos n\phi)\cos m\phi}{(\cosh\xi-\cos\phi)^2}d\phi$$

$$=2\pi\sum_{m=1}^\infty\sum_{n=1}^\infty m a_m a_n\left[\int_0^\pi\frac{\cos(n+m)\phi+\cos(n-m)\phi}{(\cosh\xi-\cos\phi)^2}d\phi-2e^{-n\xi}\int_0^\pi\frac{\cos m\phi}{(\cosh\xi-\cos\phi)^2}d\phi\right]$$

Thus, from (8), in (1) we find that

$$\left(1-\frac{1}{2\sigma}\right)^2\frac{2\sigma^2}{\pi}\int_0^1\int_0^1\frac{Z'(\ell)Z'(k)}{(\sigma-\ell)^2}\ln\left|\frac{\sigma-k}{k-\ell}\right|dk\,d\ell$$

$$=\frac{\pi\cosh^2\xi}{\sinh^3\xi}\sum_{m=1}^\infty\sum_{n=1}^\infty m a_m a_n\Big\{\left[(n+m)\sinh\xi+\cosh\xi\right]e^{-(n+m)\xi}$$

$$+\left[|n-m|\sinh\xi+\cosh\xi\right]e^{-|n-m|\xi}-2\left[m\sinh\xi+\cosh\xi\right]e^{-(n+m)\xi}\Big\}$$

$$=\pi\coth^2\xi\sum_{m=1}^\infty\sum_{n=1}^\infty m a_m a_n\left[(n-m-\coth\xi)e^{-(n+m)\xi}+(|n-m|+\coth\xi)e^{-|n-m|\xi}\right] \tag{10}$$

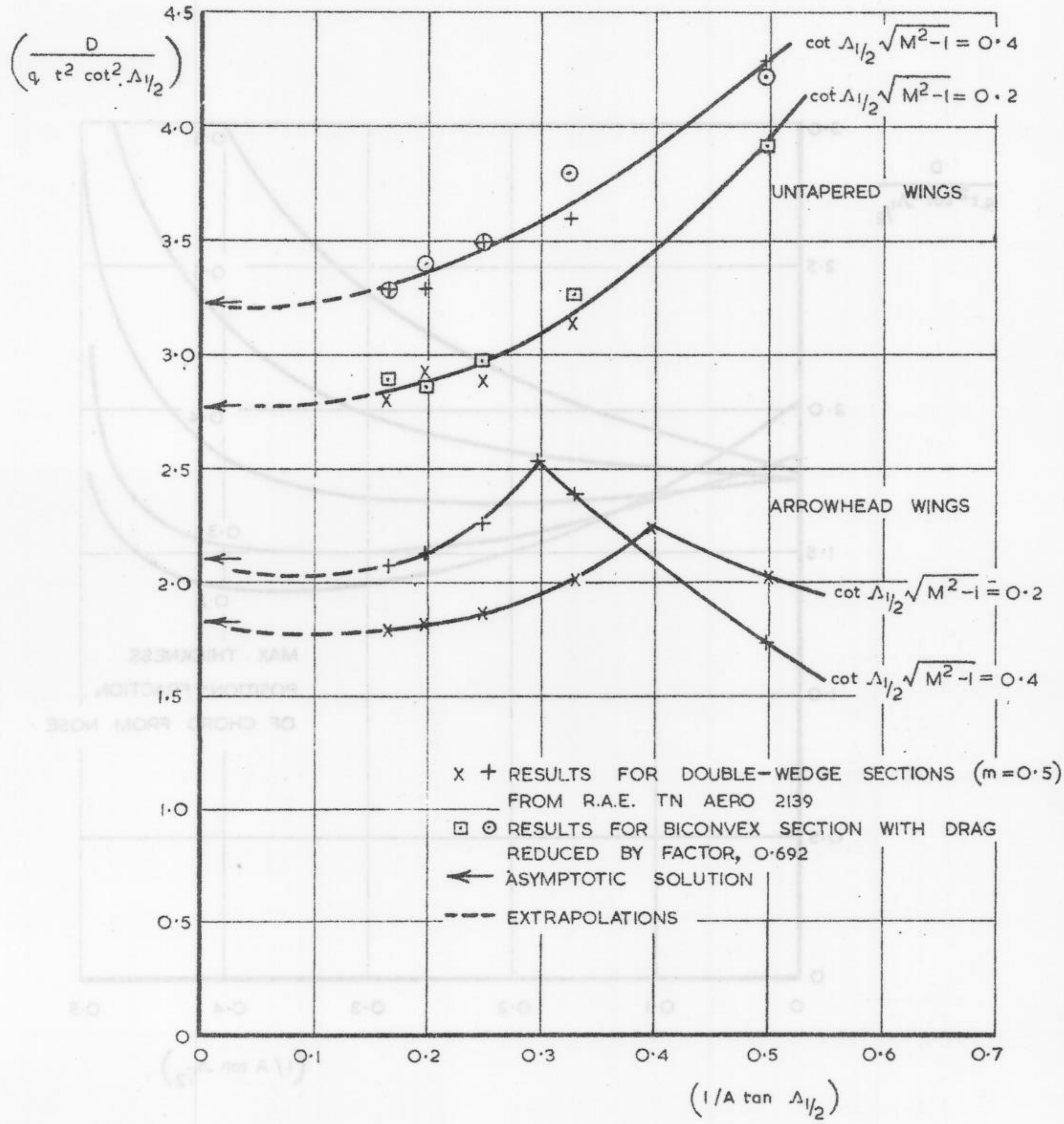
Again, from (2) and (7)

$$\int_0^1\frac{Z'(k)}{(\sigma-k)}dk=-2\sum_{n=1}^\infty na_n\int_0^\pi\frac{\cos n\theta}{\cosh\xi-\cos\theta}d\theta=-\frac{2\pi}{\sinh\xi}\sum_{n=1}^\infty na_n e^{-n\xi}$$

so that in (1),

$$\left(1-\frac{1}{2\sigma}\right)^2\frac{\sigma^2}{\pi}\left(\int_0^1\frac{Z'(k)}{(\sigma-k)}dk\right)^2=\pi\coth^2\xi\sum_{p=2}^\infty\sum_{m=1}^{p-1}m(p-m)a_m a_{p-m}\exp(-p\xi). \tag{11}$$

The results of equations (10) and (11), substituted in (1) give the required answer.

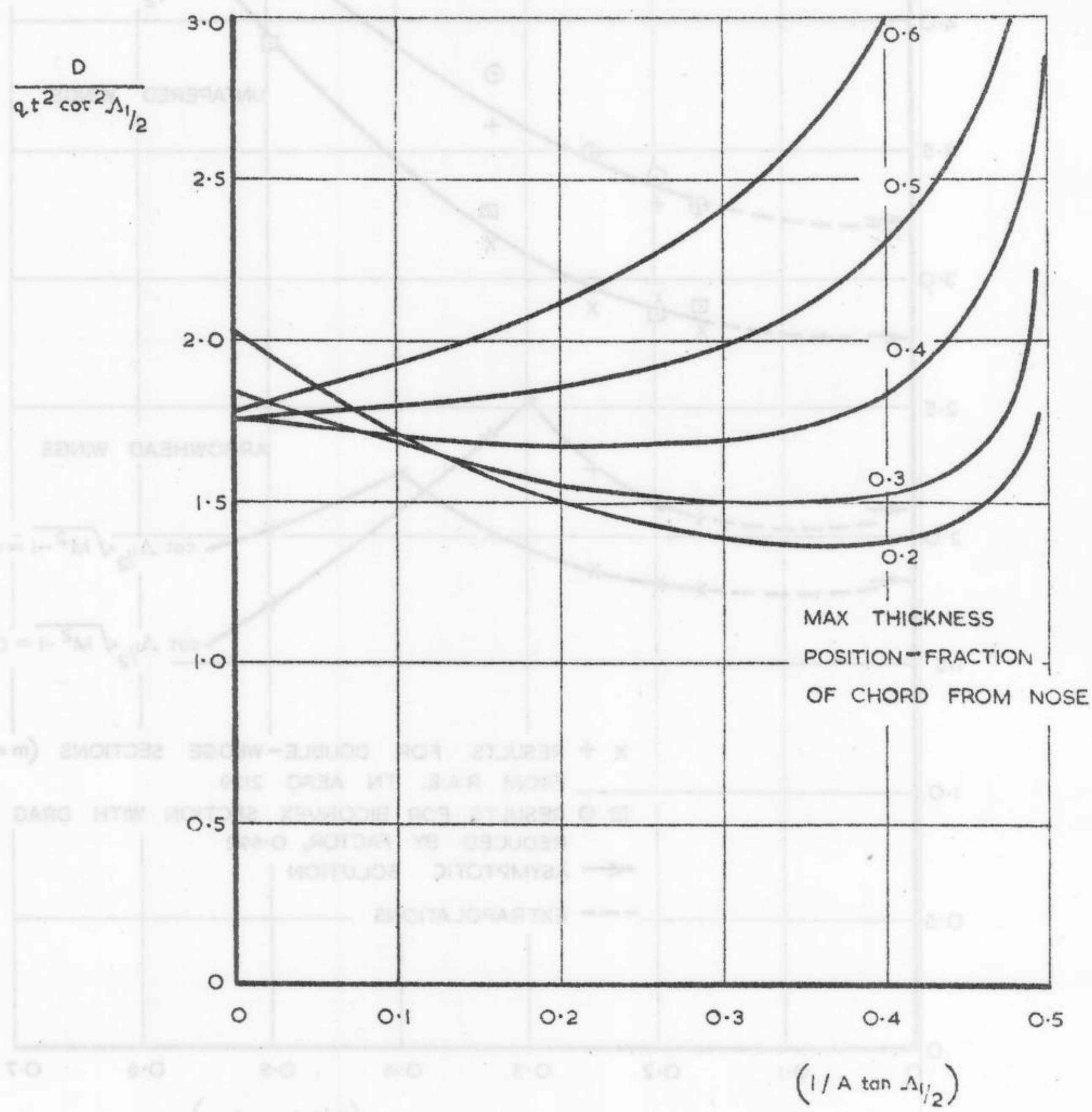


D = DRAG, q = DYNAMIC HEAD, t = ROOT MAXIMUM THICKNESS, A = ASPECT RATIO,

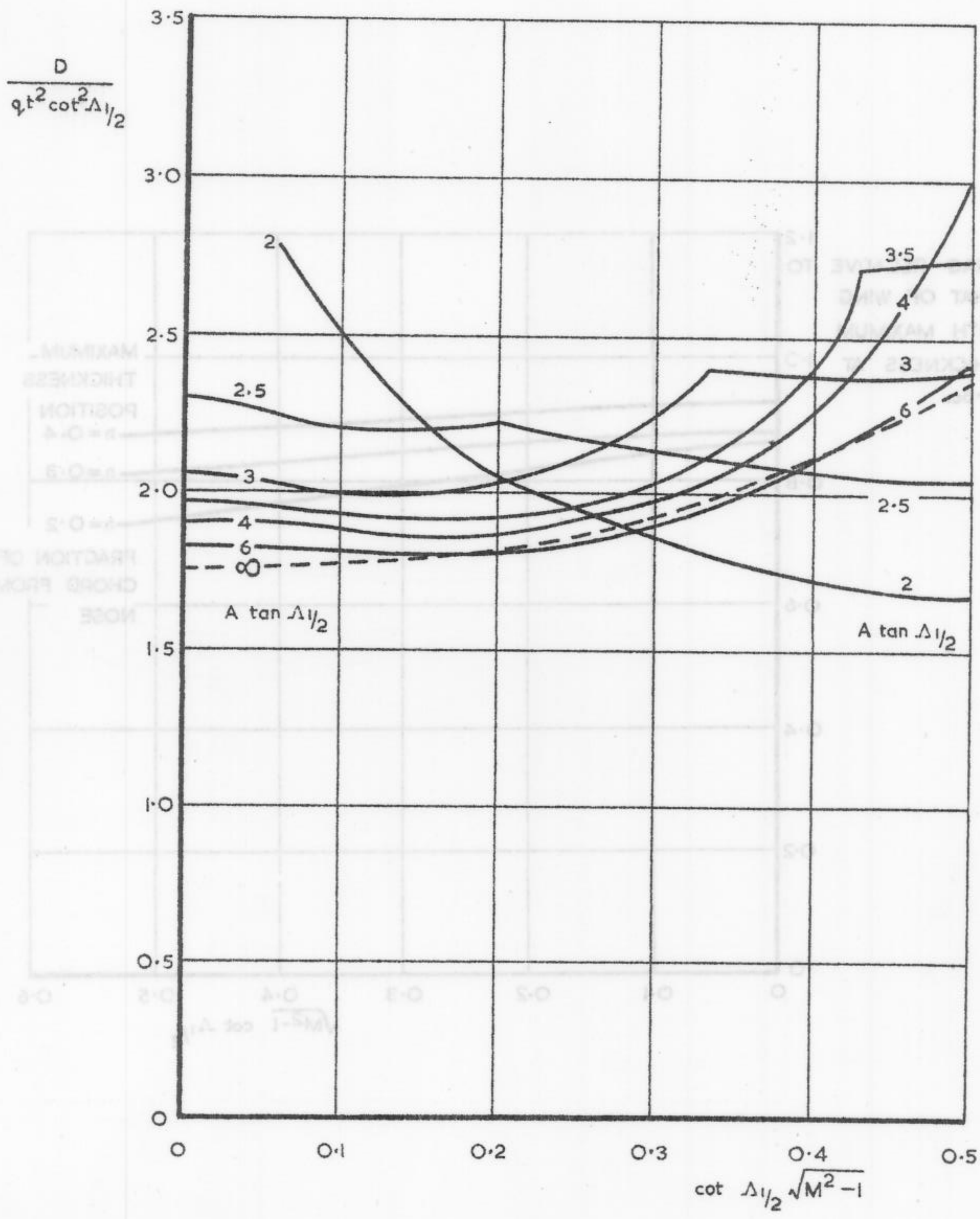
$\Lambda_{1/2}$ = HALF-CHORD SWEEPBACK ANGLE.

VARIATION OF DRAG WITH ASPECT RATIO FOR WINGS OF HIGH ASPECT RATIO

FIG. 2.

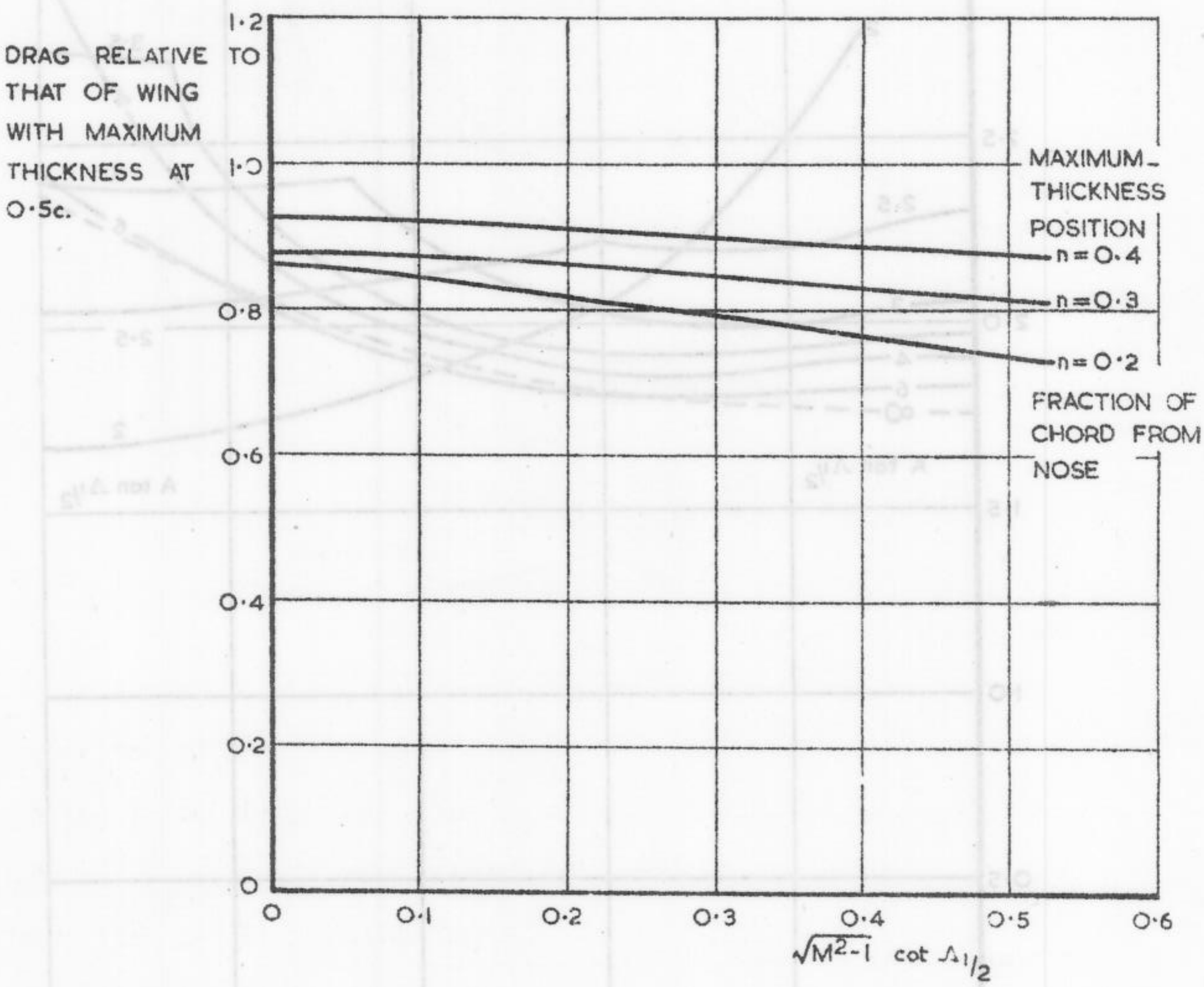


DRAG OF SLENDER WINGS OF DOUBLE — WEDGE SECTION

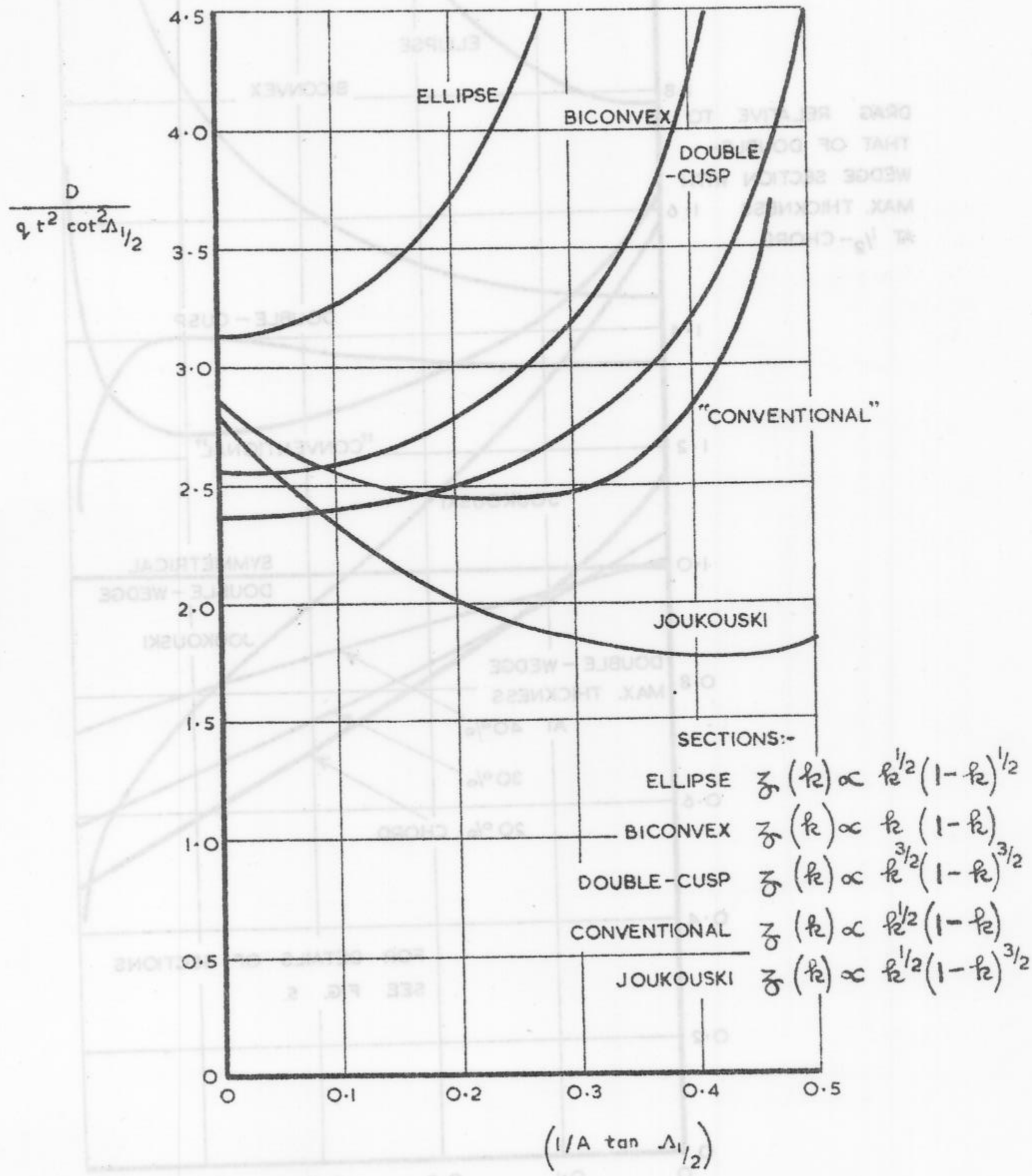


DRAG OF WINGS WITH DOUBLE-WEDGE SECTION
(MAXIMUM THICKNESS AT 0·5 CHORD)

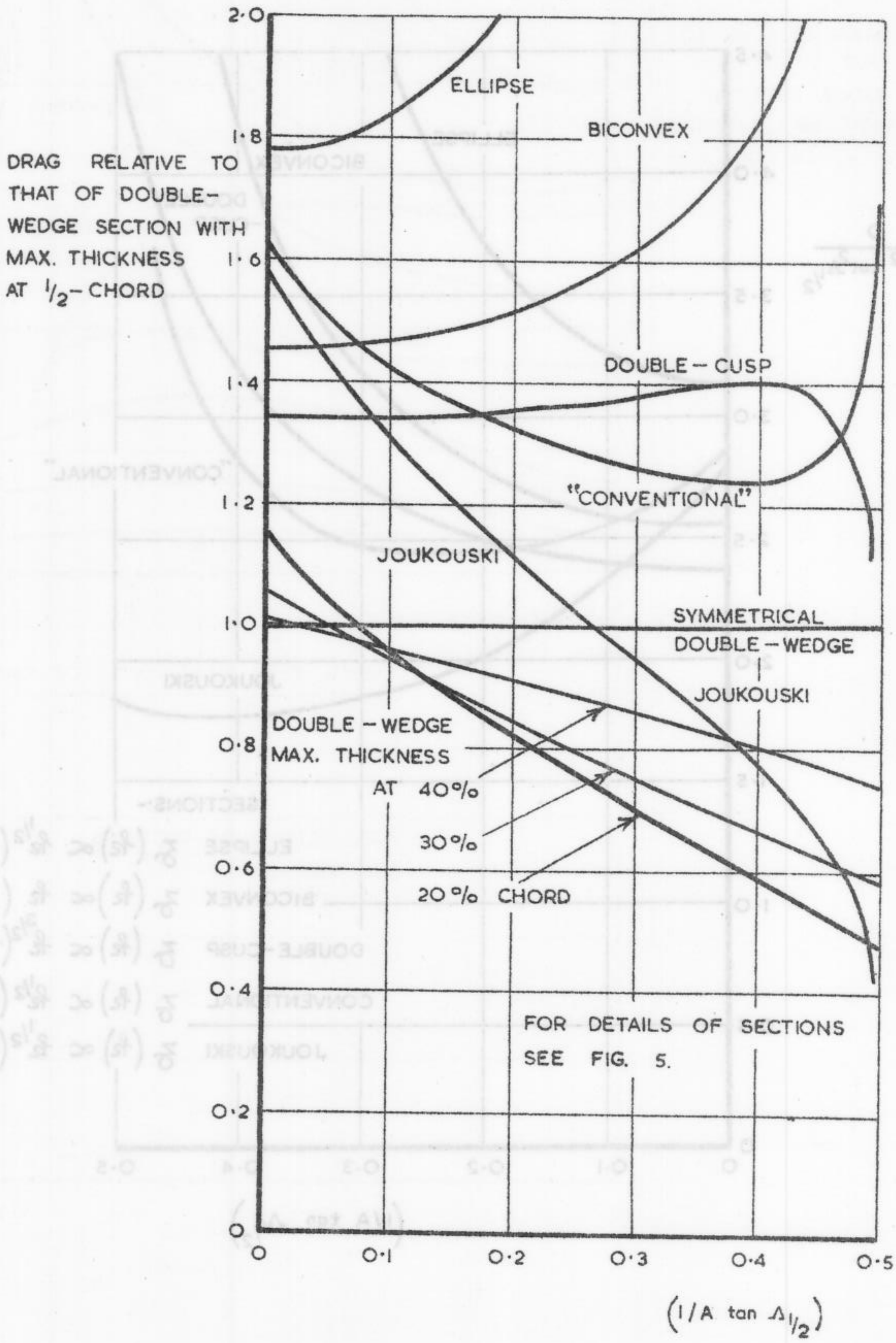
FIG. 4.



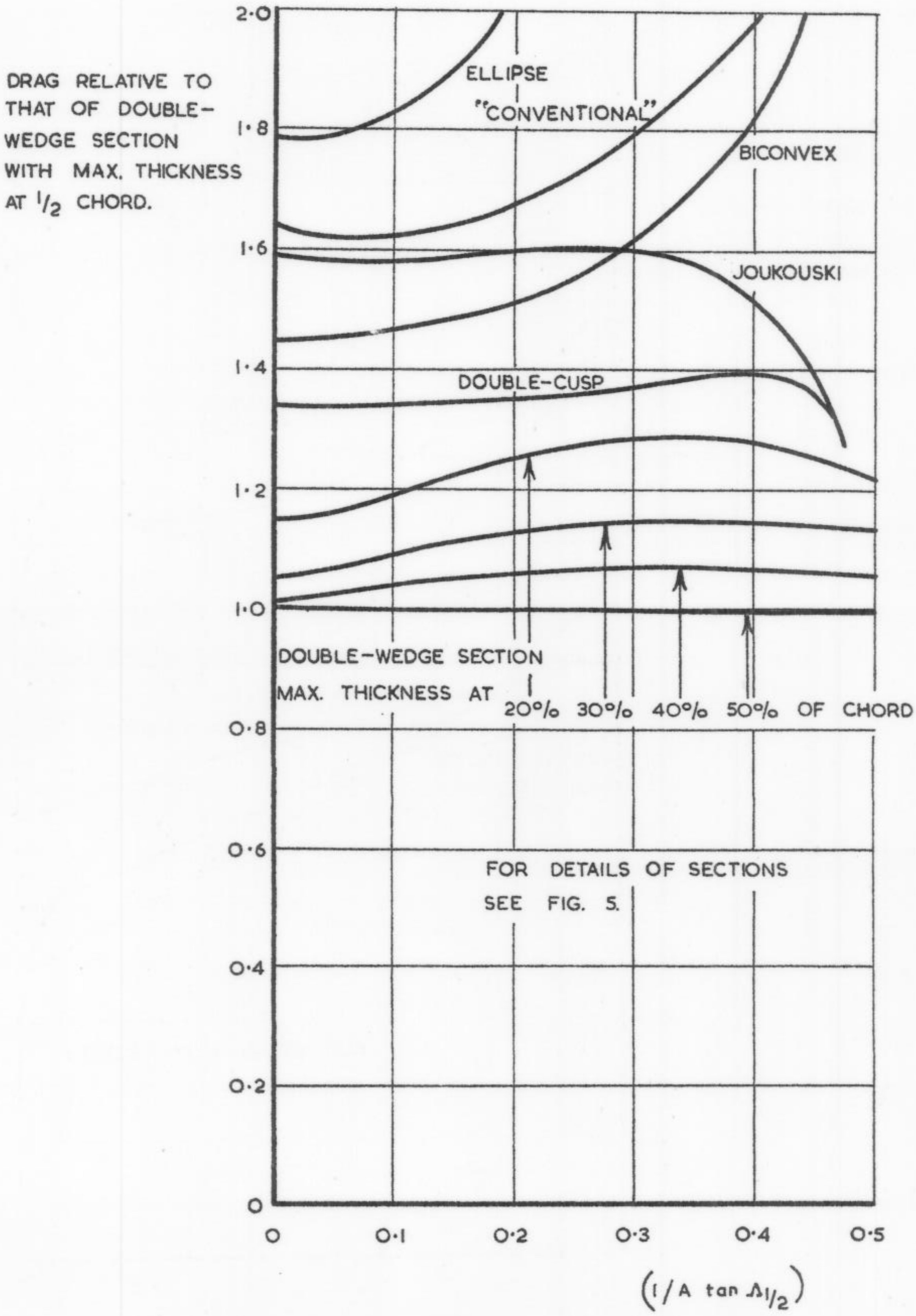
CHANGE IN RELATIVE DRAG OF DOUBLE-WEDGE SECTIONS WITH MACH NUMBER PARAMETER



DRAG OF SLENDER WINGS OF VARIOUS ELEMENTARY SECTIONS



RELATIVE DRAG OF VARIOUS WING SECTIONS ON A SLENDER WING, WITH SAME HALF CHORD SWEEP AND MAXIMUM WING THICKNESS



RELATIVE DRAG OF VARIOUS WING SECTIONS ON A SLENDER WING WITH SAME SWEEP ON MAX. THICKNESS LINE AND SAME WING THICKNESS AT ROOT